1 The induction format

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For each theorem, envision it as falling into cases that are parametrized by some integer. Imagine how larger cases can be proven from knowledge of smaller cases. Induction breaks one proof into two smaller proofs: the Base Case and the Inductive Step. This exercise asks you to state what is needed to prove for each part.

Don't do the proofs, just tell me what you need to prove for each piece. In fact, some of the statements are wrong!

The first few are done for you. See if you can understand the pattern and fill in the missing ones.

1. If T is a tree with at least one vertex, then it has one fewer edge than it has vertices.
   (a) **What are we inducting on?** The number of vertices.
   (b) **Base Case:** Prove that the tree on 1 vertex has 0 edges.
   (c) **Inductive Step (weak):** Let $n > 1$. Suppose any tree with $n-1$ vertices has $n-2$ edges. Prove that any tree with n vertices has $n-1$ edges.
   (d) **Inductive Step (strong):** Let $n > 1$. Suppose any tree with $1 \leq k < n$ vertices has $k-1$ edges. Prove that any tree with n vertices has $n-1$ edges.
2. Every positive integer is even or odd.
   (a) **What are we inducting on?** The integer.
   (b) **Base Case:** Prove that the number 1 is even or odd.
   (c) **Inductive Step (weak):** Let $n > 1$. Suppose $n-1$ is even or odd. Prove n is even or odd.
   (d) **Inductive Step (strong):** Let $n > 1$. Suppose all integers $1 \leq k < n$ are even or odd. Prove n is even or odd.
3. Any positive number of hamsters is tasty as a treat.
   (a) **What are we inducting on?** The number of hamsters.
   (b) **Base Case:** Prove that one hamster~~s~~ is tasty as a treat.
   (c) **Inductive Step (weak):** Let $n > 1$. Suppose $n-1$ hamsters is tasty as a treat. Prove that n hamsters are tasty as a treat.
   (d) **Inductive Step (strong):** Let $n > 1$. Suppose any positive number of hamsters fewer than n is tasty as a treat. Prove that n hamsters are tasty as a treat.

4. Any non-negative integer can be written as a sum of four squares.

   (a) **What are we inducting on?** the non-negative integer

   (b) **Base Case:** Prove that 0 is a sum of 4 squares.

   (c) **Inductive Step (weak):** Suppose $n > 0$.
   Suppose $n-1$ can be written as a sum of 4 squares.
   Prove that n can be written as a sum of 4 squares.

   (d) **Inductive Step (strong):** Suppose $n > 0$.
   Suppose every $0 \le k < n$ can be written as a sum of 4 squares.
   Prove that n can be written as a sum of 4 squares

5. Any non-empty finite set of hamsters is enough hamsters to warm your heart.

   (a) **What are we inducting on?** The cardinality of the set of hamsters.
   The number of hamsters.

   (b) **Base Case:** Prove that any set of one hamster is enough.

   (c) **Inductive Step (weak):** Let $n > 1$.
   Suppose any set of $n-1$ hamsters is enough.
   Prove that any set of n hamsters is enough.

   (d) **Inductive Step (strong):** Let $n > 1$.
   Suppose any set S of hamsters with $0 \le |S| \le n-1$ is enough.
   Prove that any set of n hamsters is enough.

6. Any tree on $n \geq 2$ vertices has at least 2 leaves.

   (a) **What are we inducting on?**

   (b) **Base Case:**

   (c) **Inductive Step (weak):**

   (d) **Inductive Step (strong):**

7. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for any positive integer n.

   (a) **What are we inducting on?** The integer n.

   (b) **Base Case:** Prove that $1 = \frac{1(1+1)}{2}$.

   (c) **Inductive Step (weak):** Let $n > 1$.
   Suppose $1 + \cdots + (n-1) = \frac{(n-1)(n)}{2}$.
   Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

   (d) **Inductive Step (strong):** Let $n > 1$
   Suppose $1 + \cdots + k = \frac{k(k+2)}{2}$ for all $1 \leq k < n$.
   Prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.