A function is bijective if and only if has an inverse

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Definition 1. *Let $f : A \to B$. We say that f is* surjective *if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. We say that f is* injective *if whenever $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$, then $a_1 = a_2$. We say that f is* bijective *if it is both injective and surjective.*

Definition 2. *Let $f : A \to B$. A function $g : B \to A$ is* the inverse of f *if $f \circ g = 1_B$ and $g \circ f = 1_A$.*

Theorem 1. *Let $f : A \to B$ be bijective. Then f has an inverse.*

Proof. Let $f : A \to B$ be bijective. We will define a function $f^{-1} : B \to A$ as follows. Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Let $f^{-1}(b) = a$. Since f is injective, this a is unique, so f^{-1} is well-defined.

Now we much check that f^{-1} is the inverse of f. First we will show that $f^{-1} \circ f = 1_A$. Let $a \in A$. Let $b = f(a)$. Then, by definition, $f^{-1}(b) = a$. Then $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$.

Now we will show that $f \circ f^{-1} = 1_B$. Let $b \in B$. Let $a = f^{-1}(b)$. Then, by definition, $f(a) = b$. Then $f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$. □

Theorem 2. *Let $f : A \to B$ have an inverse. Then f is bijective.*

Proof. Let $f : A \to B$ have an inverse $f^{-1} : B \to A$.

First, we will show that f is surjective. Suppose $b \in B$. Let $a = f^{-1}(b)$. Then $f(a) = f(f^{-1}(b)) = f \circ f^{-1}(b) = 1_B(b) = b$. So f is surjective.

Now, we will show that f is injective. Let $a_1, a_2 \in A$ be such that $f(a_1) = f(a_2)$. We will show $a_1 = a_2$. Let $b = f(a_1)$. Let $a = f^{-1}(b)$. Then

$$\begin{aligned} a_2 &= 1_A(a_2) \\ &= f^{-1} \circ f(a_2) \\ &= f^{-1}(f(a_2)) \\ &= f^{-1}(b) \\ &= a. \end{aligned}$$

But at the same time,

$$\begin{aligned} a_1 &= 1_A(a_1) \\ &= f^{-1} \circ f(a_1) \\ &= f^{-1}(f(a_1)) \\ &= f^{-1}(f(a_2)) \\ &= f^{-1}(b) \\ &= a. \end{aligned}$$

Therefore $a_1 = a_2$ and we have shown that f is injective. □