A function is bijective if and only if it has an inverse

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Definition 1. *Let* $f : A \to B$*. We say that* f *is* surjective *if for all* $b \in B$*, there exists an* $a \in A$ *such that* $f(a) = b$*. We say that* f *is* injective *if whenever* $f(a_1) = f(a_2)$ *for some* $a_1, a_2 \in A$*, then* $a_1 = a_2$*. We say that* f *is* bijective *if it is both injective and surjective.*

Definition 2. *Let* $f : A \to B$ *and* $g : B \to C$*. Then the* composition of f and g *is the function* $g \circ f : A \to C$ *given by* $g \circ f(a) = g(f(a))$ *for every* $a \in A$*.*

Definition 3. *Given a set* A*, the* identity function *on* A *is the function* $1_A : A \to A$ *given by* $f(a) = a$ *for all* $a \in A$*.*

Definition 4. *Let* $f : A \to B$*. A function* $g : B \to A$ *is* the inverse of f *if* $f \circ g = 1_B$ *and* $g \circ f = 1_A$*. In that case, we write* $g = f^{-1}$*.*

Theorem 1. *Let* $f : A \to B$*. Then* f *is bijective if and only if it has an inverse.*

Proof. First, let us prove the forward direction.

Let $f : A \to B$ be bijective. We will define a function $f^{-1} : B \to A$ as follows.

Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Since f is injective, this a is unique, so we may let $f^{-1}(b) = a$, and this f^{-1} is well-defined.

Now we much check that f^{-1} is the inverse of f.

First we will show that $f^{-1} \circ f = 1_A$.

Let $a \in A$. Let $b = f(a)$. Then, by definition, $f^{-1}(b) = a$. Then $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$.

Now we will show that $f \circ f^{-1} = 1_B$.

Let $b \in B$. Let $a = f^{-1}(b)$. Then, by definition, $f(a) = b$. Then $f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$.

Next, let us prove the reverse direction.

Let $f : A \to B$ have an inverse $f^{-1} : B \to A$.

First, we will show that f is surjective.

Suppose $b \in B$. Let $a = f^{-1}(b)$. Then $f(a) = f(f^{-1}(b)) = f \circ f^{-1}(b) = 1_B(b) = b$. So f is surjective.

Now, we will show that f is injective.

Let $a_1, a_2 \in A$ be such that $f(a_1) = f(a_2)$. We will show $a_1 = a_2$. Let us apply f^{-1} to the equation above:

$$\begin{aligned}
f(a_1) &= f(a_2) \\
f^{-1}(f(a_1)) &= f^{-1}(f(a_2)) \\
f^{-1} \circ f(a_1) &= f^{-1} \circ f(a_2) \\
1_A(a_1) &= 1_A(a_2) \\
a_1 &= a_2
\end{aligned}$$

Therefore $a_1 = a_2$ and we have shown that f is injective.

□