

A function is bijective if and only if it has an inverse

April 9, 2018

Definition 1. Let $f : A \rightarrow B$. We say that f is surjective if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. We say that f is injective if whenever $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$, then $a_1 = a_2$. We say that f is bijective if it is both injective and surjective.

Definition 2. Let $f : A \rightarrow B$ and $g : B \rightarrow C$. Then the composition of f and g is the function $g \circ f : A \rightarrow C$ given by $g \circ f(a) = g(f(a))$ for every $a \in A$.

Definition 3. Given a set A , the identity function on A is the function $1_A : A \rightarrow A$ given by $f(a) = a$ for all $a \in A$.

Definition 4. Let $f : A \rightarrow B$. A function $g : B \rightarrow A$ is the inverse of f if $f \circ g = 1_B$ and $g \circ f = 1_A$. In that case, we write $g = f^{-1}$.

Theorem 1. Let $f : A \rightarrow B$. Then f is bijective if and only if it has an inverse.

Proof. First, let us prove the forward direction.

Let $f : A \rightarrow B$ be bijective. We will define a function $f^{-1} : B \rightarrow A$ as follows.

Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$. Since f is injective, this a is unique, so we may let $f^{-1}(b) = a$, and this f^{-1} is well-defined.

Now we much check that f^{-1} is the inverse of f .

First we will show that $f^{-1} \circ f = 1_A$.

Let $a \in A$. Let $b = f(a)$. Then, by definition, $f^{-1}(b) = a$. Then $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$.

Now we will show that $f \circ f^{-1} = 1_B$.

Let $b \in B$. Let $a = f^{-1}(b)$. Then, by definition, $f(a) = b$. Then $f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$.

Next, let us prove the reverse direction.

Let $f : A \rightarrow B$ have an inverse $f^{-1} : B \rightarrow A$.

First, we will show that f is surjective.

Suppose $b \in B$. Let $a = f^{-1}(b)$. Then $f(a) = f(f^{-1}(b)) = f \circ f^{-1}(b) = 1_B(b) = b$. So f is surjective.

Now, we will show that f is injective.

Let $a_1, a_2 \in A$ be such that $f(a_1) = f(a_2)$. We will show $a_1 = a_2$. Let us apply f^{-1} to the equation above:

$$\begin{aligned} f(a_1) &= f(a_2) \\ f^{-1}(f(a_1)) &= f^{-1}(f(a_2)) \\ f^{-1} \circ f(a_1) &= f^{-1} \circ f(a_2) \\ 1_A(a_1) &= 1_A(a_2) \\ a_1 &= a_2 \end{aligned}$$

Therefore $a_1 = a_2$ and we have shown that f is injective. □