A function is bijective if and only if it has an inverse

March 19, 2020

Definition 1. *Let $f : A \to B$. We say that f is* surjective *if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. We say that f is* injective *if whenever $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$, then $a_1 = a_2$. We say that f is* bijective *if it is both injective and surjective.*

Definition 2. *Let $f : A \to B$ and $g : B \to C$. Then the* composition of f and g *is the function $g \circ f : A \to C$ given by $g \circ f(a) = g(f(a))$ for every $a \in A$.*

Definition 3. *Given a set A, the* identity function *on A is the function $1_A : A \to A$ given by $f(a) = a$ for all $a \in A$.*

Definition 4. *Let $f : A \to B$. A function $g : B \to A$ is* the inverse of f *if $f \circ g = 1_B$ and $g \circ f = 1_A$. In that case, we write $g = f^{-1}$.*

Theorem 1. *Let $f : A \to B$. Then f is bijective if and only if it has an inverse.*

Proof. First, let us prove the forward direction.

Let $f : A \to B$ be bijective. We will define a function $f^{-1} : B \to A$ as follows.

Now we much check that f^{-1} is the inverse of f.

First we will show that $f^{-1} \circ f = 1_A$.

Now we will show that $f \circ f^{-1} = 1_B$.

Next, let us prove the reverse direction.

Let $f : A \to B$ have an inverse $f^{-1} : B \to A$.

First, we will show that f is surjective.

Now, we will show that f is injective.

$\square$