A function is bijective if and only if it has an inverse

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Definition 1. Let $f : A \to B$. We say that $f$ is surjective if for all $b \in B$, there exists an $a \in A$ such that $f(a) = b$. We say that $f$ is injective if whenever $f(a_1) = f(a_2)$ for some $a_1, a_2 \in A$, then $a_1 = a_2$. We say that $f$ is bijective if it is both injective and surjective.

Definition 2. Let $f : A \to B$ and $g : B \to C$. Then the composition of $f$ and $g$ is the function $g \circ f : A \to C$ given by $g \circ f(a) = g(f(a))$ for every $a \in A$.

Definition 3. Given a set $A$, the identity function on $A$ is the function $1_A : A \to A$ given by $f(a) = a$ for all $a \in A$.

Definition 4. Let $f : A \to B$. A function $g : B \to A$ is the inverse of $f$ if $f \circ g = 1_B$ and $g \circ f = 1_A$. In that case, we write $g = f^{-1}$.

Theorem 1. Let $f : A \to B$. Then $f$ is bijective if and only if it has an inverse.

Proof. First, let us prove the forward direction.

Let $f : A \to B$ be bijective. We will define a function $f^{-1} : B \to A$ as follows.

Now we must check that $f^{-1}$ is the inverse of $f$.

First we will show that $f^{-1} \circ f = 1_A$.

Now we will show that $f \circ f^{-1} = 1_B$.

Next, let us prove the reverse direction.

Let $f : A \to B$ have an inverse $f^{-1} : B \to A$.

First, we will show that $f$ is surjective.

Now, we will show that $f$ is injective.