Handshake Lemma

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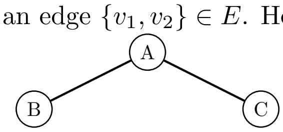
In class, we discussed the friendship problem, and the Four-Colour Theorem, and agreed that both problems were in essence problems about *graphs*, i.e. collections of objects (vertices / people / countries) for which each pair is either related in some way (joined by an edge / friends / share a border), or not. We used our language of sets to put together a definition:

Definition 1. *A graph G is an ordered pair $G = (V, E)$ where V is a set whose elements are called* vertices*, and $E \subset \mathscr{P}(V)$ is a set whose elements are called* edges*, and for which any $e \in E$ satisfies $|e| = 2$.*

We draw a graph by drawing one point for each vertex and connecting two vertices v_1 and v_2 by a line if there is an edge $\{v_1, v_2\} \in E$. Here, as an example, is the graph $G = (V = \{A, B, C\}, E = \{\{A, B\}, \{A, C\}\})$:



We further defined one more term:

Definition 2. *The number of edges containing a vertex v is said to be the* degree *of v, written* $\deg(v)$*.*

In the example above, vertex A has degree 2 and the other vertices have degree 1.

Then we proved a theorem:

Theorem 1. *In any graph G, the number of vertices of odd degree is even.*

Proof. We consider each edge of the graph G to be composed of two *half-edges* (one 'half-edge' attached to each of the vertices).

Then, we count the number of half-edges in the graph, in two different ways.

First count. The number of half-edges in the graph is a sum over all vertices of the number of half edges touching each vertex, i.e. the degree of that vertex. In other words,

$$\# \text{ of half edges} = \sum_{v \in V} \deg(v).$$

Second count. The number of half-edges is twice the number of edges. In other words,

$$\# \text{ of half-edges} = 2|E|.$$

Therefore $\sum_{v \in V} \deg(v)$ is even. If a sum is even, then the number of odd summands must be even. Hence the number of vertices of odd degree is even, as required. □