In class, we discussed the friendship problem, and the Four-Colour Theorem, and agreed that both problems were in essence problems about graphs, i.e. collections of objects (vertices / people / countries) for which each pair is either related in some way (joined by an edge / friends / share a border), or not. We used our language of sets to put together a definition:

**Definition 1.** A graph $G$ is an ordered pair $G = (V, E)$ where $V$ is a set whose elements are called vertices, and $E \subseteq \mathcal{P}(V)$ is a set whose elements are called edges, and for which any $e \in E$ satisfies $|e| = 2$.

We draw a graph by drawing one point for each vertex and connecting two vertices $v_1$ and $v_2$ by a line if there is an edge $\{v_1, v_2\} \in E$. Here, as an example, is the graph $G = (V = \{A, B, C\}, E = \{\{A, B\}, \{A, C\}\})$:

```
B   C
  \  |
   \|
A
```

We further defined one more term:

**Definition 2.** The number of edges containing a vertex $v$ is said to be the **degree** of $v$, written $\deg(v)$.

In the example above, vertex $A$ has degree 2 and the other vertices have degree 1.

Then we proved a theorem:

**Theorem 1.** In any graph $G$, the number of vertices of odd degree is even.

**Proof.** We consider each edge of the graph $G$ to be composed of two half-edges (one ‘half-edge’ attached to each of the vertices).

Then, we count the number of half-edges in the graph, in two different ways.

**First count.** The number of half-edges in the graph is a sum over all vertices of the number of half edges touching each vertex, i.e. the degree of that vertex. In other words,

$$\text{# of half edges} = \sum_{v \in V} \deg(v).$$

**Second count.** The number of half-edges is twice the number of edges. In other words,

$$\text{# of half-edges} = 2|E|.$$ 

Therefore $\sum_{v \in V} \deg(v)$ is even. If a sum is even, then the number of odd summands must be even. Hence the number of vertices of odd degree is even, as required.