Consider the following game. The board consists of \( n \) tokens. At each turn, a player must take either 1 or 2 tokens off the board (which are discarded). The player who takes the last token(s), leaving 0 tokens, is the winner.

**Definition 1.** A board with \( n \) tokens is called a *winning position* if the first player to move can always win the game, no matter how his opponent moves. Otherwise it is called a *losing position*.

In other words, \( n \) is a winning position if the first player, provided he is smart enough to make all the right moves, can always win, no matter what the second player does. It is a losing position if the second player can, by a judicious choice of moves, win the game, no matter what the first player does.

**Fact 1.** If the first player can leave the second player with a losing position, then the current position is winning.

**Fact 2.** If the first player has no choice but to leave the second player with a winning position, then the current position is losing.

**Theorem 1.** Let \( n \) be a positive integer. If \( n \) is divisible by 3, then \( n \) is a losing position. Otherwise \( n \) is a winning position.

**Proof.** Suppose \( n = 1 \) or \( n = 2 \). Then the first player can take all the tokens and win immediately. Therefore \( n \) is a winning position.

Suppose \( n = 3 \). No matter how many tokens the first player takes, the second player is left with either 1 or 2 tokens, both of which are winning positions. Therefore \( n \) is a losing position.

Suppose \( n = 4 \) or \( n = 5 \). Then the first player can take all but 3 tokens, leaving his opponent with a losing position. Therefore \( n \) is a winning position.

Suppose \( n = 6 \). No matter how many tokens the first player takes, the second player is left with either 4 or 5 tokens, both of which are winning positions. Therefore \( n \) is a losing position.

Suppose \( n = 7 \) or \( n = 8 \). Then the first player can take all but 6 tokens, leaving his opponent with a losing position. Therefore \( n \) is a winning position.

Suppose \( n = 9 \). No matter how many tokens the first player takes, the second player is left with either 7 or 8 tokens, both of which are winning positions. Therefore \( n \) is a losing position.

Suppose \( n = 10 \) or \( n = 11 \). Then the first player can take all but 9 tokens, leaving his opponent with a losing position. Therefore \( n \) is a winning position.

Suppose \( n = 12 \). No matter how many tokens the first player takes, the second player is left with either 10 or 11 tokens, both of which are winning positions. Therefore \( n \) is a losing position.

Continue forever....
Proof. We prove this by induction.

**Base Case:** Suppose \( n = 1 \) or \( n = 2 \).

Then the first player can take all the tokens and win immediately. Therefore \( n \) is a winning position.

Suppose \( n = 3 \). No matter how many tokens the first player takes, the second player is left with either 1 or 2 tokens, both of which are winning positions. Therefore \( n \) is a losing position.

**Inductive Step:** Suppose that for all positive \( k < n \), \( k \) is a winning position if and only if it is not divisible by 3. We will show that \( n \) is a winning position if and only if it is not divisible by 3.

Suppose \( n \) is not divisible by 3. Then \( n = 3\ell + k \) for some integers \( \ell \) and \( k \) such that \( \ell \geq 0 \) and \( k \in \{1,2\} \). Then the first player can take \( k \) tokens, leaving his opponent with \( 3\ell < n \) tokens, which is a losing position by the inductive hypothesis. Therefore \( n \) is a winning position.

Suppose \( n \) is divisible by 3. No matter how many tokens the first player takes, the second player is left with a number that is not divisible by 3, which is a winning position by the inductive hypothesis. Therefore \( n \) is a losing position.

**INDUCTIVE PROOF TEMPLATE**

**Theorem 2.** For all positive integers \( n \), \( P(n) \) is true.

**Proof.** We will prove \( P(n) \) is true by induction.

**Base Case:** Suppose \( n = 1 \). Note: sometimes more.

Insert a proof that \( P(n) \) is true (under the assumption \( n = 1 \)).

**Inductive Step:** Suppose that for all positive \( k < n \), \( P(k) \) is true.

Insert a proof that \( P(n) \) is true (under the assumption that \( P(k) \) is true for all \( k < n \)).

**UNROLLING AN INDUCTIVE PROOF**

**Theorem 3.** \( P(4) \) is true.

**Proof.** \( P(1) \) is true by the Base Case.

\( P(2) \) is true by the Inductive Step, since \( P(1) \) is true.

\( P(3) \) is true by the Inductive Step, since \( P(2) \) and \( P(1) \) are true.

\( P(4) \) is true by the Inductive Step, since \( P(3) \), \( P(2) \) and \( P(1) \) are true.