Worksheet on Functions

November 9, 2016

1 Functions

A function $f : A \to B$ is a way to assign one value of B to each value of A. A is the *domain*. B is the *codomain*.

More formally, you could say f is a subset of $A \times B$ which contains, for each $a \in A$, exactly one ordered pair with first element a.

2 Ways to draw a function

For each function, do the following:

1. list it as a table,
2. list it as a set of ordered pairs from $A \times B$,
3. draw it as a 'graph,'
4. draw it as an arrow diagram.

1. $f : \{a, b, c\} \to \{1, 2\}$ given by $f(a) = f(b) = 1$ and $f(c) = 2$.

2. $f : \{0, 1, 2\} \to \{0, 1, 2\}$ given by $f(x) = x$.

3. $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 1$.

3 Injective, Surjective, Bijective

Definition 1. *1. A function $f : A \to B$ is* surjective *if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (Another word for surjective is* onto.*)*

2. A function $f : A \to B$ is injective *if for every pair $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. (Another word for injective is* 1-to-1.*)*

3. A function $f : A \to B$ is bijective *if it is both surjective and injective.*

For each function on the last page, indicate if it is injective, surjective and/or bijective.

Definition 2. *The* range *of $f : A \to B$ is*

$$\{b \in B : \exists a \in A, f(a) = b\}.$$

In other words, the range is the collection of values of B that get 'hit' by the function.

1. List all functions $f : \{a, b\} \to \{x, y\}$. For each, indicate if it is injective, surjective and/or bijective. State the range.

2. Let $\mathbb{Z}_5$ denote the integers modulo 5. Let $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ be given by $f(x) = 2x \pmod 5$. List the function values out completely. Is this injective, surjective, and/or bijective? State the range.

3. Is the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x$ injective, surjective and/or bijective? Give the range in set builder notation.

4. Let $f : \mathbb{Z}_5 \to \mathbb{Z}_5$ be given by $f(x) = 5x \pmod 5$. List the function values out completely. Is this injective, surjective, and/or bijective? State the range.

5. Given two sets A and B, does there always exist a function between them?

6. Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ (like sin, x^2 etc.) which are:
   (a) injective but not surjective
   (b) surjective but not injective
   (c) bijective
   (d) neither injective nor surjective

7. Explain the properties of the graph of a function $f : \mathbb{R} \to \mathbb{R}$ in the plane $\mathbb{R}^2$ which correspond to injectivity or surjectivity (e.g. vertical line test).

8. Let $\mathbb{R}^+$ denote the positive real numbers. Write a nice proof that the function $f : \mathbb{R} \to \mathbb{R}^+$ given by $f(x) = e^x$ is bijective.

9. If $|A| = n$ and $|B| = m$, how many different functions are there $f : A \to B$ (in terms of n and m)? How many of these are bijective? How many of these are injective? How many are surjective?