Worksheet on Functions – Solutions

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1 Functions

A function $f : A \to B$ is a way to assign one value of B to each value of A. A is the *domain*. B is the *codomain*.

More formally, you could say f is a subset of $A \times B$ which contains, for each $a \in A$, exactly one ordered pair with first element a.

2 Ways to draw a function

For each function, do the following:

1. list it as a table,
2. list it as a set of ordered pairs from $A \times B$,
3. draw it as a 'graph,'
4. draw it as an arrow diagram.

1. $f : \{a, b, c\} \to \{1, 2\}$ given by $f(a) = f(b) = 1$ and $f(c) = 2$.

2. $f : \{0, 1, 2\} \to \{0, 1, 2\}$ given by $f(x) = x$.

3. $f : \mathbb{N} \to \mathbb{N}$ given by $f(x) = x + 1$.

3 Injective, Surjective, Bijective

Definition 1. *1. A function $f : A \to B$ is* surjective *if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (Another word for surjective is* onto*.)*

2. A function $f : A \to B$ is injective *if for every pair $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. (Another word for injective is* 1-to-1*.)*

3. A function $f : A \to B$ is bijective *if it is both surjective and injective.*

For each function on the last page, indicate if it is injective, surjective and/or bijective.

Definition 2. *The* range *of $f : A \to B$ is*

$$\{b \in B : \exists a \in A, f(a) = b\}.$$

In other words, the range is the collection of values of B that get 'hit' by the function.

1. List all functions $f : \{a, b\} \to \{x, y\}$. For each, indicate if it is injective, surjective and/or bijective. State the range.

   A function is just a rule telling where a and b go:

in	out
a	x
b	x

   This is neither injective nor surjective, hence not bijective. The range is $\{x\}$.

in	out
a	x
b	y

   This is injective and surjective, hence bijective. The range is $\{x, y\}$.

in	out
a	y
b	x

   This is injective and surjective, hence bijective. The range is $\{x, y\}$.

in	out
a	y
b	y

   This is neither injective nor surjective, hence not bijective. The range is $\{y\}$.

   Note that there are four possible functions by multiplication principle, because to specify a function, you have to answer two binary questions:

   (a) Where does a go? (two choices)

   (b) Where does b go? (two choices)

2. Is the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x$ injective, surjective and/or bijective? Give the range in set builder notation.

   This function is injective, since whenever $f(x_1) = f(x_2)$, then $2x_1 = 2x_2$, hence $x_1 = x_2$.

   This function is not surjective, since there is no x for which $f(x) = 1$.

   Therefore the function is not bijective.

   The range is the even integers, i.e. $\{2x : x \in \mathbb{Z}\}$.

3. Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ (like sin, x^2 etc.) which are:

   (a) injective but not surjective
   (b) surjective but not injective
   (c) bijective
   (d) neither injective nor surjective

   See the video for some graphs (which is where you can really see whether it is injective, surjective or bijective), but briefly, here are some examples that work (there are many more correct answers):

   (a) injective but not surjective: $f(x) = e^x$.
   (b) surjective but not injective: $f(x) = (x-1)x(x+1)$.
   (c) bijective: $f(x) = x$.
   (d) neither injective nor surjective: $f(x) = \sin(x)$.

4. Explain the properties of the graph of a function $f : \mathbb{R} \to \mathbb{R}$ in the plane $\mathbb{R}^2$ which correspond to injectivity or surjectivity (e.g. vertical line test).

   To test injectivity is to test whether it happens that two different input x values result in the same output y values (if there are no such collisions, it is injective). So a function is injective if it passes a *at-most-one horizontal line test*, i.e. any horizontal line passes through the graph at most once.

   To test surjectivity is to test whether all y values get 'hit' by the function. So a function is surjective if it passes an *at-least-one horizontal line test*, i.e. any horizontal line passes through the graph at least once.

   Notice the nice symmetry in these interpretations. A function is bijective if every horizontal line passes through the graph exactly once.

5. Let $\mathbb{R}^+$ denote the positive real numbers. Write a nice proof that the function $f : \mathbb{R} \to \mathbb{R}^+$ given by $f(x) = e^x$ is bijective.

   Let $f : \mathbb{R} \to \mathbb{R}^+$ be given by $f(x) = e^x$.

   First, we will show that f is surjective. Let $y \in \mathbb{R}^+$, that is, y is a positive real number. We must find an $x \in \mathbb{R}$ so that $f(x) = y$. It will suffice to take $x = \log y$, which is defined since y is positive. For, then $f(x) = f(\log y) = e^{\log y} = y$ as required.

   Next, we will show that f is injective. Suppose that $f(x_1) = f(x_2)$ for some $x_1, x_2 \in \mathbb{R}$. Then

$$
\begin{aligned}
e^{x_1} &= e^{x_2} \\
\log(e^{x_1}) &= \log(e^{x_2}) \\
x_1 &= x_2
\end{aligned}
$$

   . Therefore $x_1 = x_2$.