1 Functions

A function \( f : A \to B \) is a way to assign one value of \( B \) to each value of \( A \). \( A \) is the domain. \( B \) is the codomain.

More formally, you could say \( f \) is a subset of \( A \times B \) which contains, for each \( a \in A \), exactly one ordered pair with first element \( a \).

2 Ways to draw a function

For each function, do the following:

1. list it as a table,
2. list it as a set of ordered pairs from \( A \times B \),
3. draw it as a ‘graph,’
4. draw it as an arrow diagram.

1. \( f : \{a, b, c\} \to \{1, 2\} \) given by \( f(a) = f(b) = 1 \) and \( f(c) = 2 \).

2. \( f : \{0, 1, 2\} \to \{0, 1, 2\} \) given by \( f(x) = x \).

3. \( f : \mathbb{N} \to \mathbb{N} \) given by \( f(x) = x + 1 \).
3 Injective, Surjective, Bijective

Definition 1. 1. A function $f : A \to B$ is surjective if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (Another word for surjective is onto.)

2. A function $f : A \to B$ is injective if for every pair $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. (Another word for injective is 1-to-1.)

3. A function $f : A \to B$ is bijective if it is both surjective and injective.

For each function on the last page, indicate if it is injective, surjective and/or bijective.

Definition 2. The range of $f : A \to B$ is

$$\{b \in B : \exists a \in A, f(a) = b\}.$$ 

In other words, the range is the collection of values of $B$ that get ‘hit’ by the function.

1. List all functions $f : \{a, b\} \to \{x, y\}$. For each, indicate if it is injective, surjective and/or bijective. State the range.

2. Is the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x$ injective, surjective and/or bijective? Give the range in set builder notation.
3. Give examples of functions $f : \mathbb{R} \to \mathbb{R}$ (like $\sin$, $x^2$ etc.) which are:

(a) injective but not surjective  
(b) surjective but not injective  
(c) bijective  
(d) neither injective nor surjective  

4. Explain the properties of the graph of a function $f : \mathbb{R} \to \mathbb{R}$ in the plane $\mathbb{R}^2$ which correspond to injectivity or surjectivity (e.g. vertical line test).  

5. Let $\mathbb{R}^+$ denote the positive real numbers. Write a nice proof that the function $f : \mathbb{R} \to \mathbb{R}^+$ given by $f(x) = e^x$ is bijective.