Worksheet on Functions

November 14, 2016

1 Functions: terminology

- A function $f : A \to B$ is a way to assign one value of B to each value of A. A is the *domain*. B is the *codomain*.
- The function $f : A \to A$ that takes $f(a) = a$ for every $a \in A$ has a special name: *the identity function*.
- The *image* of a subset $X \subset A$ is the set of things X goes to in B, i.e.

$$f(X) = \{f(x) : x \in X\}.$$

- The image of A has a special name, the *range*.
- The *pre-image* of a subset $Y \subset B$ is the set of things that go into Y, i.e.

$$f^{-1}(Y) = \{a \in A : f(a) \in Y\}.$$

- A function $f : A \to B$ is *surjective* if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. (Another word for surjective is *onto*.)
- A function $f : A \to B$ is *injective* if for every pair $a_1, a_2 \in A$, $a_1 \neq a_2$ implies $f(a_1) \neq f(a_2)$. (Another word for injective is 1-*to*-1.)
- A function $f : A \to B$ is *bijective* if it is both surjective and injective.

2 Practice problems

1. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^2$.
   (a) What is the domain?
   (b) What is the codomain?
   (c) What is the range?
   (d) Is this the identity function? (Yes/No)
   (e) Is this injective?
   (f) Is this surjective?
   (g) Is this bijective?
   (h) What $f([0, 1])$?
   (i) What is $f^{-1}([4, \infty])$?
2. Let $f : \mathbb{Z} \to \mathbb{R}$ be given by $f(x) = 3x$.
   (a) What is the domain?
   (b) What is the codomain?
   (c) What is the range? (use set builder)
   (d) Is this the identity function? (Yes/No)

(e) Is this injective?
(f) Is this surjective?
(g) Is this bijective?
(h) What $f(\{x \in \mathbb{Z} : |x| < 3\})$?
(i) What is $f^{-1}(5\mathbb{Z})$? Note that $5\mathbb{Z}$ is notation for $5\mathbb{Z} = \{5x : x \in \mathbb{Z}\}$.

3. Draw a graph of a function which is injective but not surjective, which has domain and codomain $\mathbb{R}$, and satisfies $f([0, \infty]) = [1, \infty]$ and $f^{-1}([0, \infty]) = \mathbb{R}$.

4. Given two sets of equal cardinality $|A| = |B| = n$.
   (a) How many functions are there $f : A \to B$?
   (b) How many of these are bijective?
   (c) Can you construct one which is injective but not bijective?

5. Let A and B be finite sets, and suppose $f : A \to B$.
   Fill in the table with P (possible) and I (impossible).

	$\|A\| = \|B\|$	$\|A\| > \|B\|$	$\|A\| < \|B\|$
bijective			
surjective, not injective			
injective, not surjective			
neither injective nor surjective			

6. Prove that the function $f : \mathbb{Z} \to \mathbb{Z}$ given by $2x + 3$ is injective, but not surjective.

7. Use the notation $\mathbb{Z}/n\mathbb{Z}$ for the residues modulo n (so that $\mathbb{Z}/n\mathbb{Z}$ has size n). For which a is the function $f : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$ given by $x \mapsto ax$ bijective?