# Example Combinatorial Proof

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**Theorem 1.** *For all non-negative $n, k \in \mathbb{Z}$,*

$$\binom{k}{k} + \binom{k+1}{k} + \binom{k+2}{k} + \cdots + \binom{n}{k} = \binom{n+1}{k+1}$$

*Proof.* We will show that both sides of the equation count the number of $(k+1)$-element subsets of $\{1, 2, \ldots, n+1\}$.

The right hand side of the equation counts this by definition.

The left hand side counts this by conditioning on the largest element in the subset, as follows.

Any subset has a largest element $\ell$ satisfying $k+1 \leq \ell \leq n+1$. To count the number of subsets having largest element $\ell$, we must count the number of ways to choose the remaining $k-1$ elements (i.e. besides the largest) from $\{1, \ldots, \ell-1\}$. There are $\binom{\ell-1}{k-1}$ ways to do this, hence $\binom{\ell-1}{k-1}$ subsets of $\{1, 2, \ldots, n+1\}$ having $\ell$ as largest element.

Therefore, the total number of $(k+1)$-element subsets of $\{1, 2, \ldots, n+1\}$ is obtained by summing the number of such subsets with largest element $\ell$ for each $k+1 \leq \ell \leq n+1$. We obtain

$$\sum_{\ell=k+1}^{n+1} \binom{\ell-1}{k-1},$$

which is exactly the left hand side of the equation. $\square$