Example Combinatorial Proofs

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Theorem 1. *For all $n \geq k \geq 0$,*

$$\binom{n}{k} = \binom{n}{n-k}$$

Illustration: Subsets of size 2 from $S = \{a, b, c, d, e\}$. ($k = 2$, $n = 5$)

Subset	k elements chosen	$n - k$ elements not chosen
{a,b}	a,b	c,d,e
{a,c}	a,c	b,d,e
{a,d}	a,d	b,c,e
{a,e}	a,e	b,c,d
{b,c}	b,c	a,d,e
{b,d}	b,d	a,c,e
{b,e}	b,e	a,c,d
{c,d}	c,d	a,b,e
{c,e}	c,e	a,b,d
{d,e}	d,e	a,b,c

Proof. We will show that both sides of the equation count the number of ways to choose a subset of size k from a set of size n.

The left hand side of the equation counts this by definition.

Now we consider the right hand side. To choose a subset of size k, we can instead choose the $n - k$ elements to exclude from the subset. There are $\binom{n}{n-k}$ ways to do this. Therefore the right hand side also counts the desired quantity. □

Theorem 2. *For all $n \geq k \geq 1$,*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Illustration: Subsets of size 3 from $S = \{a, b, c, d, e\}$ ($k = 3$, $n = 5$).

Subsets	Subsets containing a	Subsets not containing a
{a,b,c}	{a,b,c}	
{a,b,d}	{a,b,d}	
{a,b,e}	{a,b,e}	
{a,c,d}	{a,c,d}	
{a,c,e}	{a,c,e}	
{a,d,e}	{a,d,e}	
{b,c,d}		{b,c,d}
{b,c,e}		{b,c,e}
{b,d,e}		{b,d,e}
{c,d,e}		{c,d,e}
$10 = \binom{5}{3}$	$6 = \binom{4}{2}$	$4 = \binom{4}{3}$

Proof. I have broken the proof under three headings to highlight its structure.

1. **QUESTION:** We will show that both sides of the equation count the number of ways to choose a subset of size k from a set S of size n.

2. **LEFT:** The left hand side of the equation counts this by definition.

3. **RIGHT:** Let $s \in S$ be a fixed element. We will show that the right hand side counts the desired quantity by conditioning on whether s is in the subset.

   First, we will count how many subsets of size k include s. Since such a subset includes s, there are $k - 1$ other elements in the subset, which must be chosen from the remaining $n - 1$ elements of S. Therefore there are $\binom{n-1}{k-1}$ such subsets.

   Second, we will count how many subsets of size k do not include s. Since the subset does not include s, all of its k elements are chosen from the remaining $n - 1$ elements of S. Therefore there are $\binom{n-1}{k}$ such subsets.

   Since any subset of size k either includes s or does not (but not both), the total number of subsets is the sum of the counts in the two cases.

□

Theorem 3. *For all $n \geq k \geq m \geq 0$,*

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}.$$

Proof.

1. **QUESTION:** We will show that both sides of the equation count the number of ways to choose a committee of k students from a student body of n students, where, in addition, a subcomittee of m of the k students form the executive committee.

2. **LEFT:** We will describe the counting process.

   (a) First, we choose k students from the student body of n students, to form the committee. There are $\binom{n}{k}$ ways to do this.

   (b) Then we choose m students from among those k to form the subcommittee. There are $\binom{k}{m}$ ways to do this.

   By the multiplication principle, the left hand side counts the desired quantity.

3. **RIGHT:** We will describe the counting process.

   (a) First, we choose m students from the student body of n students, to form the executive committee. There are $\binom{n}{m}$ ways to do this.

   (b) Then we choose $k - m$ of the remaining portion of the student body (which consists of $n - m$ students), to form the non-executive part of the committee. There are $\binom{n-m}{k-m}$ ways to do this.

   By the multiplication principle, the right hand side counts the desired quantity.

$\square$

Theorem 4. *For all $n \geq 1$,*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Proof.

1. **QUESTION:** We will show that both sides of the equation count the number of ways to choose a subset of a set S of n elements.

2. **RIGHT:** When creating a subset of S, for each element of S, there are two options: to include it or not to include it. Since we make this choice n times (once for each element), there are a total of 2^n possible sequences of choices. Each sequence gives exactly one subset, and every subset results from exactly one sequence. Therefore there are a total of 2^n subsets of S. Therefore the right hand side counts the desired quantity.

3. **LEFT:** We will show that the left hand side counts the desired quantity by conditioning on the size of the subset. The possible sizes of subsets of S are $0 \leq k \leq n$. By definition, there are $\binom{n}{k}$ subsets of size k. Therefore the total number of subsets is the sum on the left hand side.

□

Theorem 5. *For all $n \geq 1$,*

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

Proof.

1. **QUESTION:** We will show that both sides of the equation count the number of ways to choose a *non-empty* subset of the set $S = \{1, 2, \ldots, n\}$.

2. **RIGHT:** As in the last proof, the number of subsets of S is 2^n. Exactly one of these is empty, so there are $2^n - 1$ non-empty subsets.

3. **LEFT:** We will show that the left hand side counts the desired quantity by conditioning on the largest element of the subset. Every non-empty subset has a largest element k where $1 \leq k \leq n$.

   Let $1 \leq k \leq n$. We will count the number of subsets of S having largest element k. Such a subset includes k and does not include $k+1, \ldots, n$. Therefore to specify such a subset we must decide $k-1$ choices: for each element of $\{1, 2, \ldots, k-1\}$, we must decide to include or not include that element. Therefore there are 2^{k-1} such subsets.

   Summing over all possible k, we see that the left hand side counts the desired quantity.

□