Worksheet on Functions and Composition

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1. For the following functions, determine which compositions are defined, and for each composition that is defined, determine the composition.

$f : \{1, 2, 3\} \to \{a, b\}$

x	$f(x)$
1	a
2	b
3	b

$g : \{a, b\} \to \{1, 2\}$

x	$g(x)$
a	1
b	2

$h : \{a, b, c\} \to \{1, 2, 3\}$

x	$h(x)$
a	1
b	1
c	3

Answer: For a composition to be defined, the codomain of the first function must match the domain of the second. Therefore the possibilites are (please note the domain and codomain of the compositions are shown):

$f \circ h : \{a, b, c\} \to \{a, b\}$

x	$f \circ h(x)$
a	a
b	a
c	b

$g \circ f : \{1, 2, 3\} \to \{1, 2\}$

x	$g \circ f(x)$
1	1
2	2
3	2

2. For each of the following functions, determine if it has an inverse and what the inverse should be. If it has no inverse, explain why. If you give an inverse, prove it (meaning, compute the composition in both orders and check it is the identity).

   (a) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 1$.
   Answer: $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = \sqrt[3]{x-1}$. We check the composition both directions:
   $$f \circ f^{-1}(x) = f(\sqrt[3]{x-1}) = (\sqrt[3]{x-1})^3 + 1 = x - 1 + 1 = x$$
   and
   $$f^{-1} \circ f(x) = f^{-1}(x^3 + 1) = \sqrt[3]{x^3 + 1 - 1} = \sqrt[3]{x^3} = x$$
   Note that this answer depends on the fact that the real cube root of a real number is a unique well-defined real number which cubes to the original. That is, we can use $\sqrt[3]{x}$ as a well-defined function in formulas.

   (b) $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$.
   Answer: This one doesn't have an inverse, because the square root isn't a unique well-defined value: there are two square roots. For example, $f(1) = 2 = f(-1)$ so, to "undo" this, should $f^{-1}(2)$ be 1 or -1? Recall that an inverse should satisfy $f^{-1} \circ f(x) = x$. But in order for $f^{-1} \circ f(1) = 1$, we need $f^{-1}(2) = 1$. But we also want $f^{-1} \circ f(-1) = -1$, so we also need $f^{-1}(2) = -1$ *also*. This is impossible.
   Takeaway: The failure of injectivity meant we can't have an inverse!

(c) $g : \{a, b, c\} \to \{1, 2, 3\}$ given by

x	$g(x)$
a	1
b	1
c	3

Answer: This one doesn't have an inverse, because if it did, then $f \circ f^{-1}(2) = 2$, but 2 isn't even in the range of f, so this can never happen.

Takeaway: The failure of surjectivity meant we can't have an inverse!

(d) $g : \{a, b, c\} \to \{1, 2, 3\}$ given by

x	$g(x)$
a	1
b	3
c	2

This one does have an inverse! It is the following:

(e) $g^{-1} : \{1, 2, 3\} \to \{a, b, c\}$ given by

x	$g^{-1}(x)$
1	a
2	c
3	b