Combinatorial Proof

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1 Counting by 'conditioning' or cases

1. How many ways can you pack your lunch if you have 50 identical hard-boiled eggs and five identical sandwiches, and you can take *either* the egg carton (which fits up to 12 eggs but no sandwiches) or the sandwich container (which fits up to two sandwiches but no eggs).
2. How many ways can you pick a subset of 8 students, if you must pick *either* a subset of size 3 or a subset of size 7?
3. How many ways can you pick a subset of $2 \leq k \leq 4$ students from a class of 20?
4. How many ways can you pick a subset of any size from a class of 20 (the answer is 2^{20} but write the answer AS A SUM similar to the last item)?
5. How many ways can you make a word (double letters not allowed) from the alphabet that is either all vowels or all consonants?
6. How many ways can you choose a non-empty subset of $\{1, 2, \ldots, n\}$ whose largest element is 3?
7. How many ways can you choose a non-empty subset of $\{1, 2, \ldots, n\}$ whose largest element is $1 \leq k \leq 3$?
8. How many ways can you choose a non-empty subset of $\{1, 2, \ldots, n\}$ whose largest element is any size? The answer is $2^n - 1$, but write the answer AS A SUM similar to the last item.

2 Combinatorial Proof

We've seen some examples of combinatorial proof now. A combinatorial proof is a proof that shows some equation is true by explaining why both sides count the same thing. Its structure should generally be:

1. Explain what we are counting.
2. Explain why the LHS counts that correctly.
3. Explain why the RHS counts that correctly.
4. Conclude that both sides are equal since they count the same thing.

The purpose of this worksheet is to write combinatorial proofs. You should work together to get well-written ones (I suggest you use the template above), and I can come around and critique them. Then we'll have a quiz where you'll have to write one.

1. Give a combinatorial proof that

$$\binom{n}{k} = \binom{n}{n-k}.$$

2. Give a combinatorial proof that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

3. Give a combinatorial proof that

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

4. From Section 1, you actually showed two theorems, write them out as combinatorial proofs now:

   (a)
$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

   (b)
$$\sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

5. More combinatorial proofs:

   (a) Actually, if you take $\binom{n}{k}$ *by definition*, as counting the number of subsets of size k of a set of size n, then showing

   $$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

   is by combinatorial proof.

   (b)
   $$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}.$$

   (c)
   $$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n-1}\binom{n}{1}+\binom{n}{n}\binom{n}{0} = \binom{2n}{n}$$

   (d) Invent your own!