# Combinatorial Proof Examples

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A combinatorial proof is a proof that shows some equation is true by explaining why both sides count the same thing. Its structure should generally be:

Explain what we are counting.

Explain why the LHS (left-hand-side) counts that correctly.

Explain why the RHS (right-hand-side) counts that correctly.

Conclude that both sides are equal since they count the same thing.

**HOW TO STUDY:** In what follows, the first few are worked completely, including a properly written proof. The next few I have given answers to the three key questoins, in brief, and you should write a complete proof. In the last few, I have given only a hint, which can be expanded first by answering the three questions, then writing a complete proof.

1. Give a combinatorial proof that

$$\binom{n}{k} = \binom{n}{n-k}.$$

   (a) What are we counting?

   We are counting how many ways a subset of $k$ things can be chosen from $n$ things.

   (b) How does the left side count this?

   The symbol $\binom{n}{k}$ counts this by definition.

   (c) How does the right side count this?

   To choose a subset of $k$ things, it is equivalent to choose $n-k$ things to exclude from the subset. There are $\binom{n}{n-k}$ ways to do this.

   Now here is a complete theorem and proof.

   **Theorem 1.** *Suppose $n \geq 1$ is an integer. Suppose $k$ is an integer such that $1 \leq k \leq n$. Then*

$$\binom{n}{k} = \binom{n}{n-k}.$$

   *Proof.* We will explain that both sides of the equation count the number of ways to choose a subset of $k$ things from $n$ things (and they must therefore be equal).

   The left side counts this by definition.

   To choose a subset of $k$ things, it is equivalent to choose $n-k$ things to exclude from the subset. There are $\binom{n}{n-k}$ ways to do this. Therefore the right side also counts this.

   Hence both sides are equal. □

2. Give a combinatorial proof that

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

(a) What are we counting?

The number of ways to choose a subset of size $k$ from a set of size $n$.

(b) How does the left side count this?

By definition.

(c) How does the right side count this?

Choose an element from the set of size $n$, and call it $x$. We condition on whether $x$ is in the chosen subset.

Case 1: If $x$ is to be included in the chosen subset, then there are $\binom{n-1}{k-1}$ ways to complete the subset.

Case 2: If $x$ is to be excluded from the chosen subset, then there are $\binom{n-1}{k}$ ways to complete the subset.

Adding the counts from the two cases gives the total number of ways to choose the subset.

Here is a complete theorem and proof.

**Theorem 2.** *Suppose $n \geq 1$ is an integer. Suppose $k$ is an integer such that $1 \leq k \leq n$. Then*

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

*Proof.* We will demonstrate that both sides count the number of ways to choose a subset of size $k$ from a set of size $n$.

The left hand side counts this by definition.

To see that the right hand side also counts this, choose an element from the set of size $n$, and call it $x$. We condition on whether $x$ is in the chosen subset.

Case 1: If $x$ is to be included in the chosen subset, then there are $\binom{n-1}{k-1}$ ways to complete the subset.

Case 2: If $x$ is to be excluded from the chosen subset, then there are $\binom{n-1}{k}$ ways to complete the subset.

Adding the counts from the two cases gives the total number of ways to choose the subset. This gives the right hand side of the equation.

Therefore both sides enumerate the same quantity and must therefore be equal. $\square$

3. Give a combinatorial proof that

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

(a) What are we counting? Chaired committees.

(b) How does the left side count this? Choose the committee, then the chair.

(c) How does the right side count this? Choose the chair, then the rest of the committee.

4. Give a combinatorial proof that

$$\binom{n}{k}\binom{k}{m} = \binom{n}{m}\binom{n-m}{k-m}$$

(a) What are we counting? Committees of size $k$ having a head council of size $m$. Or $A \subseteq B \subseteq C$, where $|A| = m$, $|B| = k$, $|C| = n$.

(b) How does the left side count this? Choose $B$ first, then $A$.

(c) How does the right side count this? Choose $A$ first, then the rest of $B$.

5.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

(a) What are we counting? Subset of a set of size $n$.

(b) How does the left side count this? Condition on the size of the subset.

(c) How does the right side count this? Determine the subset by asking each element of $n$ whether it is in or not.

6.

$$\sum_{k=0}^{n-1} 2^k = 2^n - 1.$$

(a) What are we counting? Non-empty subsets of $\{1, 2, 3, \ldots, n\}$.

(b) How does the left side count this? Condition on the largest element of the subset.

(c) How does the right side count this? Usual power-set computation, but empty set is disallowed.

7. Actually, if you take $\binom{n}{k}$ *by definition*, as counting the number of subsets of size $k$ of a set of size $n$, then showing

$$\binom{n}{k} = \frac{n!}{k!(n-k)!}$$

is by combinatorial proof.

Hint: Explain how to choose $k$ things by ordering $k$ things, then correcting for overcounting.

8.

$$\binom{2n+2}{n+1} = \binom{2n}{n+1} + 2\binom{2n}{n} + \binom{2n}{n-1}.$$

Hint: Choose $n+1$ people from among $2n+2$ people. Condition on whether we include Mary and Kate, only one of them, or neither of them.

9.

$$\binom{n}{0}\binom{n}{n}+\binom{n}{1}\binom{n}{n-1}+\binom{n}{2}\binom{n}{n-2}+\cdots+\binom{n}{n-1}\binom{n}{1}+\binom{n}{n}\binom{n}{0} = \binom{2n}{n}$$

Hint: Choose $n$ people from among $2n$ people. Suppose $n$ are women and $n$ are men, and condition on how many women are chosen.

10. The above can also be written as

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

Hint: This is because $\binom{n}{k} = \binom{n}{n-k}$.