MATHEMATICS 2001
INDIVIDUAL WORK DUE FRIDAY FRIDAY MAR 23 AT 8 AM

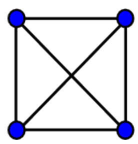
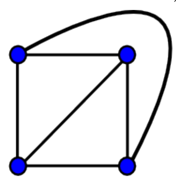
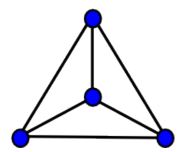
General information. You can work with others as long as you write up your solutions yourself (not copying). Hand this in on canvas as with previous groupwork.

Please spend approximately 2 hours on this homework. If you don't finish all the proofs, hand in what you have.

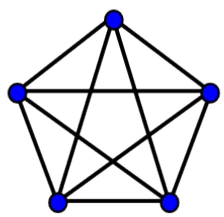
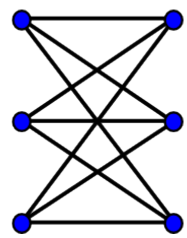
Planar graphs.

Definition 1. *A graph G is* planar *if it can be drawn in the plane in such a way that no two edges cross.*

Edges do not need to be drawn straight, and the plane is as big as you need. For example, the complete graph on 3 vertices is planar, because you can draw it as a triangle with no edges crossing. The complete graph on 4 vertices is also planar, because you *can* draw it without crossings. Here are some different ways to draw it (the first has crossings, but the second two do not):

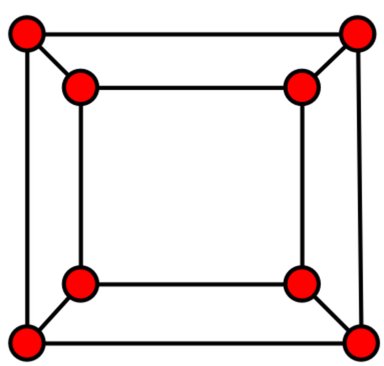
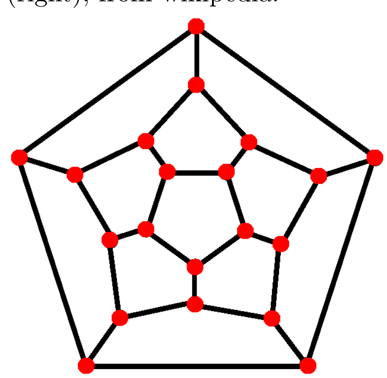
  

On the other hand, here are some graphs that you simply cannot draw without crossings. These are *non-planar.*

(These images are from http://www.boost.org/doc/libs/1_66_0/libs/graph/doc/planar_graphs.html.)

Euler's Formula. The first day of class, we considered polyhedra, counting edges, faces and vertices. The only aspect of the polyhedra that mattered to the counting question was the *graph* formed by their vertices and edges, and the number of regions these broke the surface into. So we can actually "pull open" the polyhedron (imagine pulling open one face with your hands, like peeling an orange) to create a planar graph. For example, here's the cube (left) and dodecahedron (right), from wikipedia:

The conjecture you made on day 1 is the following:

Theorem 1. *Consider any connected planar graph G with E edges, $V \geq 1$ vertices, and such that it divides the plane into F regions (including the outer region extending to infinity). Then $V - E + F = 2$.*

For example, for the cube above, we have $E = 12$, $V = 8$ and $F = 6$. For the dodecahedron, $E = 30$, $V = 20$ and $F = 12$.

Your task. Your task is to prove Theorem 1.

(1) First, check it by drawing and counting for some planar graphs.
(2) Then, prove it by induction. (Do not look it up online!)
(3) Finally, explain why we need "planar" in the theorem statement.

Note: it is possible to induct on the number of vertices, number of edges, or number of faces. Some options are more complicated than others. Some may require you to redefine graphs as possibly involving edges connecting a vertex to itself. Etc. Your challenge is to come up with a *completely correct proof.* The basic idea may be simple, but the details may be devilish. Make sure you think it through very carefully.

You may work with others, but do not look up a proof online. Generate your own.