MATHEMATICS 2001
GROUPWORK DUE FRIDAY FRIDAY MAR 2 AT 8 AM

General information. You should produce two things to hand in: a groupwork report using the website template (handwritten; hand in in class), and a PDF of four proofs uploaded to canvas, which must be **typset** in some way (not handwritten).

Choose someone to fill each of these roles: leader, scribe, presenter. The leader's job is to keep everyone on task. The scribe's job is to create the two documents to hand in. The presenter's job is to present solutions in class (if time permits; not every group will present). Each week, these tasks should rotate, so everyone gets a turn.

Please spend approximately 2 hours on this homework. Work collaboratively, as you did last week. If you don't finish all the proofs, hand in what you have.

Existence Proofs. An *existence theorem* is a theorem which asserts the existence of something. Usually one proves it *constructively*, i.e. you construct an example. Here's an example:

Theorem 1. *There exist sets of each positive cardinality.*

Proof. Let n be any positive integer. Then the set $\{x \in \mathbb{Z} : 1 \leq x \leq n\}$ has cardinality n. □

In other words, we've shown any positive cardinality can be acheived by actually constructing a set of the desired cardinality.

Prove the following theorem:

Theorem 2. *There exists a graph on n vertices, having exactly $n-1$ edges.*

Non-existence Proofs. An *non-existence theorem* is a theorem which asserts the non-existence of something. Sometimes one proves it by contradiction, i.e. assume it exists and reach a contradiction. Here is an example:

Theorem 3. *There do not exist any even primes greater than* 2.

Proof. Suppose p is an even prime greater than 2. Then $p = 2k$ for some $k \in \mathbb{Z}$. Since $p > 2$, we have $k > 1$. But then p is composite, not prime, a contradiction. □

Here's a proof that avoids contradiction, but is essentially the same:

Proof. Suppose p is an even **integer** greater than 2. Then $p = 2k$ for some $k \in \mathbb{Z}$. Since $p > 2$, we have $k > 1$. Therefore p is composite. We have shown that all even integers greater than 2 are composite. □

Another example is your previous proof that there is no largest integer. There also, you assumed there was a largest integer, and reached a contradiction.

Prove the following:

Theorem 4. *There are no graphs on* 3 *vertices, having* 4 *edges.*

Hint: count the number of places you could put an edge.

The main event. Prove the following two theorems.

Definition 1. *The complete graph on n vertices is the graph on n vertices having an edge between every pair of vertices.*

Theorem 5. *The complete graph on n vertices has $n(n-1)/2$ edges.*

Definition 2. *A graph G is* connected *if there is a path between any two vertices along edges in the graph.*

Theorem 6. *Let G be a graph on n vertices such that the degree of each vertex is at least $n/2$. Then G is connected.*