MATHEMATICS 2001

GROUPWORK DUE FRIDAY FRIDAY FEB 23 AT 8 AM

General information. You should produce two things to hand in: a groupwork report using the website template (handwritten; hand in in class), and a PDF of four proofs uploaded to canvas, which must be **typset** in some way (not handwritten).

Choose someone to fill each of these roles: leader, scribe, presenter. The leader's job is to keep everyone on task. The scribe's job is to create the two documents to hand in. The presenter's job is to present solutions in class (if time permits; not every group will present). Each week, these tasks should rotate, so everyone gets a turn.

Please spend approximately 2 hours on this homework. Work collaboratively, as you did last week. If you don't finish all the proofs, hand in what you have. You may skip some to get to others you care more about.

Play a Game! I want you to play the following game with your group-mates.

The board consists of n tokens (use pencils, coins, torn up bits of paper, or whatever you have on hand). At each turn, a player must take either 1 or 2 tokens off the board (which are discarded). The player who takes the last token(s) on their turn, leaving 0 tokens, is the winner.

Example game:

- 4 tokens are on the board, and Alice takes 1
- 3 tokens remain, so Bob takes 2
- 1 token remains, so Alice takes it and wins.

Play the game, and try to figure out the winning strategy.

Definition 1. *A board with n tokens is called a* winning position *if the first player to move can always win the game, no matter how his opponent moves. Otherwise it is called a* losing position.

In other words, n is a winning position if the first player, provided he is smart enough to make all the right moves, can always win, no matter what the second player does. Otherwise it is a losing position.

First, fill in a chart of winning and losing positions:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16 ...
win?																

The following are two lemmas:

Lemma 1. *If the first player can leave the second player with a losing position, then the current position is winning.*

Lemma 2. *If the first player has no choice but to leave the second player with a winning position, then the current position is losing.*

Note: You can try to write proofs of these lemmas, if you like. Or you can think really hard about them and come to believe them. Nevertheless, they are true!

Your task for this week is to fill in the blank below correctly with a criterion about the integer n, and then to prove the resulting theorem.

Theorem 1. *Let n be a positive integer. If ______, then n is a losing position. Otherwise n is a winning position.*

Note: If you can't prove this theorem, don't just hand in a blank sheet. Do your best to do partial solutions. For example, you could prove that 9 is a losing position and 10 is a winning position. Find a way to demonstrate your progress.