MATHEMATICS 2001
GROUPWORK DUE FRIDAY FRIDAY FEB 16 AT 8 AM

**General information.** You should produce two things to hand in: a groupwork report using the website template (handwritten; hand in in class), and a PDF of four proofs uploaded to canvas, which must be **typset** in some way (not handwritten).

Choose someone to fill each of these roles: leader, scribe, presenter. The leader's job is to keep everyone on task. The scribe's job is to create the two documents to hand in. The presenter's job is to present solutions in class (if time permits; not every group will present). Each week, these tasks should rotate, so everyone gets a turn.

Please spend approximately 2 hours on this homework. Work collaboratively, as you did last week. If you don't finish all the proofs, hand in what you have. You may skip some to get to others you care more about.

**Prove it!**

**Theorem 1.** *Let $n$ and $k$ be integers. Suppose $n$ pigeons are to be placed into $k$ holes. Suppose $n > k$. Then at least one hole is shared (i.e. has more than one pigeon in it).*

Hint: Use contradiction.

**Theorem 2.** *Among any 3 positive integers, there exist two of the same parity.*

Hint: The two pigeonholes are labelled "even" and "odd". Put the three numbers into the two pigeonholes. What happens if two numbers are in the same pigeonhole?

**Theorem 3.** *Suppose* 5 *points are placed in the interior of a square with unit sides. Then some pair of these points is at distance less than or equal to* $1/\sqrt{2}$.

Hint: Yes, this is the same problem as from last week. Revisit it with pigeonhole. Break the square into four equal squares of side 1/2, and place the five points (pigeons) into the four regions (pigeonholes).

**Theorem 4.** *Suppose $a$ and $b$ have the same remainder when divided by $n$. Then $n$ divides $a - b$.*

Hint: To say that *$a$ has remainder $r$ when divided by $n$* (we only talk about $0 \leq r < n$) is to say that there exists some $k$ so that $a = nk + r$. For example, 5 has remainder 2 when divided by 3 because $5 = 3 \cdot 1 + 2$, i.e. $k = 1$ does the trick. This is a direct proof, involving practice in the methods from the first week or two.

**Theorem 5.** *Amongst any $n$ positive integers, there exist two whose difference is divisible by $n - 1$.*

Hint: This is a generalization of Theorem 2. The pigeonholes are the possible remainders when divided by $n - 1$, i.e. $0, 1, \ldots, n - 2$. What if two integers end up in the same pigeonhole?

**Theorem 6.** *Suppose $n$ people are in a room together. Suppose each pair is either a pair of friends or not. Then there are two people with the same number of friends.*

Hint: Suppose "Pigeonhole 0" contains the people who are friendless, "Pigeonhole 1" contains those people having only one friend, "Pigeonhole 2" contains those people having exactly two friends, and so on. Place the $n$ people (pigeons) into these $n$ pigeonholes according to their "friendliness". Is it really possible to have one person in each pigeonhole? Why not?

**Theorem 7.** *Any $X \subseteq \{1, 2, 3, 4, 5, 6, 7, 8\}$ such that $|X| = 5$ will include two elements $a$ and $b$ such that $a + b = 9$.*

Hint: First, try some examples. Then, think how to use pigeonhole.

**Fun Extra Problem.** Consider this magic trick: A magician asks an audience member to pick five cards, which are not shown to the magician. The magician's accomplice looks at the cards, picks four of the cards, and shows these four to the magician in an order of his choosing. The magician then correctly guesses the fifth card.

Can you figure out a mathematical way to perform this trick? If so, demonstrate it on your classmates on presentation day! Hint: The pigeonhole principle guarantees that such a trick is possible. But it is challenging to come up with a good way to do it.