MATHEMATICS 2001
GROUPWORK DUE FRIDAY FRIDAY FEB 9 AT 8 AM

General information. You should produce two things to hand in: a groupwork report using the website template (handwritten; hand in in class), and a PDF of four proofs uploaded to canvas, which must be **typset** in some way (not handwritten).

Choose someone to fill each of these roles: leader, scribe, presenter. The leader's job is to keep everyone on task. The scribe's job is to create the two documents to hand in. The presenter's job is to present solutions in class (if time permits; not every group will present). Each week, these tasks should rotate, so everyone gets a turn.

Please spend approximately 2 hours on this homework. Work collaboratively, as you did last week. If you don't finish all the proofs, hand in what you have.

It's puzzle time! The goal today is to produce some nicely-written proofs. The method of contradiction will work well for these, but it is not mandated – any proof will do. You may not use internet or external resources.

"When you have eliminated the impossible, whatever remains, however improbable, must be the truth." – Sherlock Holmes, in the novel The Sign of the Four (1890) by Sir Arthur Conan Doyle

Theorem 1. *There is no largest integer.*

Hint: Well, suppose there were... what if you doubled it or added one?

Theorem 2. *There is no smallest positive rational number.*

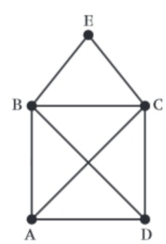
Hint: Try the largest integer proof first.

Theorem 3. *There are infinitely many primes p such that $p + 2$ is composite.*

Hint: If not, then eventually we run out of p such that $p + 2$ is composite. (Note: We may assume you and the audience know what a prime and composite number are (if not, see Hammack, p. 90), and all their standard, basic properties. I won't nitpick that aspect of it.)

Interesting fact: Mathematicians believe there are also infinitely many primes p such that $p + 2$ is prime, but no one has been able to prove it.

Theorem 4. *Consider the following picture:*



It is impossible to traverse this diagram along the edges in a loop (ending where you begin) using each edge exactly once.

Hints: This is a finite thing to check, but that is tedious (you'd have to try all possible loops!). Instead, imagine there did exist a loop, and reach a contradiction. Try considering how such a loop would appear from the perspective of one fixed vertex.

Theorem 5. *Consider a unit square with sides of length* 1*. Suppose* 5 *points are placed inside this square. Then there exist two points x and y among these five, such that the distance between x and y is less than or equal to $1/\sqrt{2}$.*

Hint: Draw the square and subdivide it into four smaller squares.

Theorem 6. *The set of sets which do not contain themselves cannot exist.*

Hint: Let S be that set. Is it true that $S \in S$? Is it false?