MATHEMATICS 2001
GROUP WORK DUE FRIDAY FRIDAY APR 27 AT 8 AM

General information. This is groupwork – you need only hand in one assignment for your group.

Please spend approximately 2 hours on this homework. If you don't finish all the proofs, hand in what you have.

Assignment

This week, the goal is to write some combinatorial proofs. That is, you should prove an equation by showing that both sides of the equation are correct answers to the same question. We have talked through a few of these in class, and the goal here is to write up beautifully written proofs of these.

Theorem 1. *For $n \geq 1$,*

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n.$$

Directions: Prove that the left hand side counts the ways to form a committee (of any size) from a class of n students. Prove that the right hand side counts the same thing. Conclude that they must be equal.

Theorem 2. *For $n \geq 0$,*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}.$$

Directions: Observe that the right side counts how many ways you can choose two different numbers from the set $S = \{1, 2, \ldots, n\}$ (it is a binomial coefficient, or by multiplication principle with some overcounting). Then show that the left hand side also counts this. To do this, break into cases based on the largest element selected. In other words, in Case k, the largest element selected is the integer $k+1$ and you must count how many ways to pick the second element (which must be smaller). Conclude that left side equals right side since they count the same thing.

Theorem 3. *For $0 \leq k \leq n$. Then,*

$$\sum_{m=k}^{n} \binom{m}{k} = \binom{n+1}{k+1}.$$

Directions: Observe that the right hand side counts subsets of size $k+1$ from the set $S = \{1, 2, \ldots, n+1\}$. Now, prove that the left hand side also counts this. Carefully check the indices (they're surprising). The idea here is to break into cases based on the largest number in the subset. So, count the number of subsets of S of size $k+1$ whose largest element is 1, whose largest element is 2 etc. Then conclude left side equals right side.

Theorem 4. *For $n \geq 1$,*

$$\sum_{k=0}^{n} 2^k = 2^{n+1} - 1.$$

Directions: This is a puzzle for you to figure out how to do with combinatorial proof.