MATHEMATICS 2001
GROUP WORK DUE FRIDAY FRIDAY APR 13 AT 8 AM

General information. This is groupwork – you need only hand in one assignment for your group.

Please spend approximately 2 hours on this homework. If you don't finish all the proofs, hand in what you have.

The Continuity Game. Here's the definition of continuity, in abstract mathematics. You will need to contend with this definition if you take Analysis. The purpose of this assignment is to give an example of strategies for approaching a complicated definition like this.

Definition 1. *Let $f : \mathbb{R} \to \mathbb{R}$. We say f is* continuous *at the point x if it satisfies the following property: for any $\epsilon > 0$, there exists a $\delta > 0$ such that for any $a \in \mathbb{R}$, if $|x - a| < \delta$ then $|f(x) - f(a)| < \epsilon$.*

Think of this as a game: no matter how close you want to guarantee $f(x)$ and $f(a)$ to be, you can choose a sufficiently close to x to attain that goal.

Play this game with the function $f(x) = 2x + 3$, as follows:

(1) Person A chooses $x \in \mathbb{R}$ (the point at which we will test continuity).
(2) Person B chooses $\epsilon > 0$. (Try $\epsilon = 1$ or $\epsilon = 1/3$ for example.)
(3) Person C chooses $\delta > 0$. (This person has the tricky job.)
(4) Person D checks whether the following statement is true: for any $a \in \mathbb{R}$, if $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.

Person D announces "True" or "False". If he announces "True", then Person C wins. If he announces "False", then Persons A and B win.

Here's a sample game:

(1) Person A chooses $x = 0$.
(2) Person B chooses $\epsilon = 1$.
(3) Person C chooses $\delta = 1/2$.
(4) Person D has to check the following statement: for any $a \in \mathbb{R}$, if $|a| < 1/2$, then $|2(0)+3-(2a+3)| < 1$. In fact, this statement is true, since $|a| < 1/2$ does indeed imply that $|2(0)+3-(2a+3)| = |2a| < 1$.
(5) Therefore Person C wins.

Can you see how person C knew how to choose δ to win?

The function f is continuous at x if no matter what B does, C can win by playing cleverly. In fact, the function $f(x) = 2x + 3$ is continuous at every x. Begin by playing the game together until each person can reliably win when they are person C. In the example above, Person C notices that taking $\delta = \epsilon/2$ will always allow him to win.

Then prove as many of the following theorems as you have time for.

Theorem 1. *The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 5$ is continuous at $x = 0$.*

Theorem 2. *The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 5$ is continuous at $x = 1$.*

Theorem 3. *The function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 3x + 5$ is continuous at all values x.*