

Theorem. Every amount of postage  $n \geq 5$  cents can be made by a combination of 2-cent and 5-cent stamps.

Pf. We will induct on  $n$ .

Base Case: 5 can be formed of 2-and 5-cent stamps since  $5 = 5$ .  
6 " " " "  
 $6 = 2 + 2 + 2$ .

Inductive Step: Let  $n > 6$ .

Suppose all amounts  $5 \leq m < n$  can be formed of 2 and 5 cent stamps.

Then  $n - 2 = 2k + 5l$  for some  $k, l \in \mathbb{N}$ , since  $5 \leq n - 2 < n$ .

So  $n = 2(k+1) + 5l$ .

So  $n$  can be formed of 2 and 5 cent stamps.



Theorem. Every amount of postage  $n \geq 5$  cents can be made by a combination of 2-cent and 5-cent stamps.

Rough work..

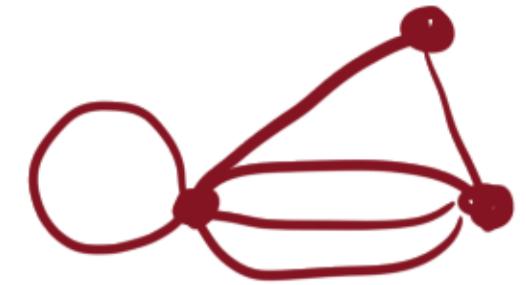
The diagram illustrates the recursive construction of postage amounts. It starts with the base case  $5 = 5$ , which is labeled "Base". This is followed by  $6 = 2 + 2 + 2$ , also labeled "Base". A green arrow labeled "+2" points from the 5-cent oval to the 6-cent oval. The next step is  $7 = 5 + 2$ , which is labeled "Ind" (inductive). Another green arrow labeled "+2" points from the 6-cent oval to the 7-cent oval. The next step is  $8 = 2 + 2 + 2 + 2$ , also labeled "Ind". A red arrow labeled "+2" points from the 7-cent oval to the 8-cent oval. The next step is  $9 = 5 + 2 + 2$ , also labeled "Ind". A red arrow labeled "+2" points from the 8-cent oval to the 9-cent oval. Finally, the equation  $10 = 5 + 5 = 2 + 2 + 2 + 2 + 2$  is shown. A green arrow labeled "-2-2" points from the 9-cent oval to the 10-cent equation, and a green arrow labeled "+5" points from the 5-cent oval to the 10-cent equation.

$$5 = 5 \quad \text{Base}$$
$$6 = 2 + 2 + 2 \quad \text{Base}$$
$$7 = 5 + 2 \quad \text{Ind}$$
$$8 = 2 + 2 + 2 + 2 \quad \text{Ind}$$
$$9 = 5 + 2 + 2 \quad \text{Ind}$$
$$10 = 5 + 5 = 2 + 2 + 2 + 2 + 2 \quad \begin{matrix} -2-2 \\ +5 \end{matrix}$$

Defn A multigraph allows multiple edges and loops.

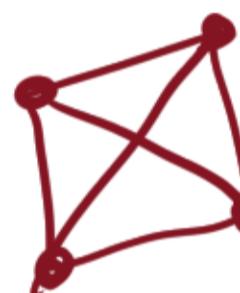


Example.

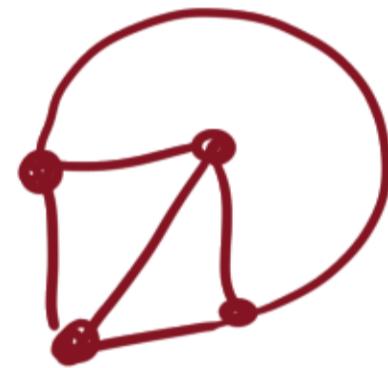


Def' A (multi)graph is planar if it can be drawn in the plane without crossing edges.

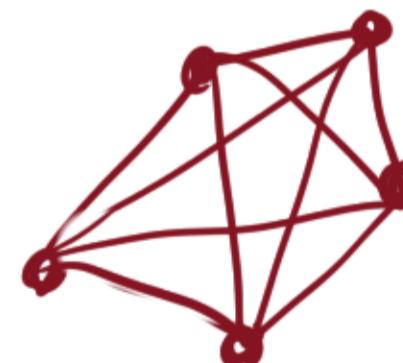
Eg.



is planar because

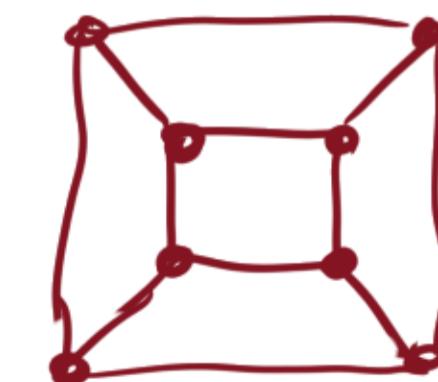
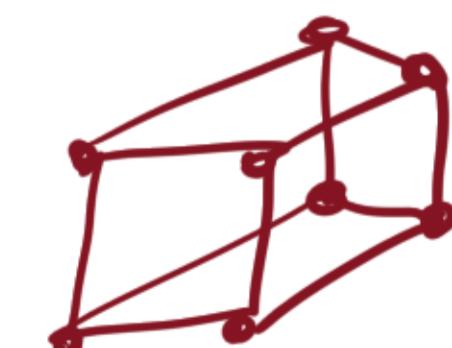


is the same graph.



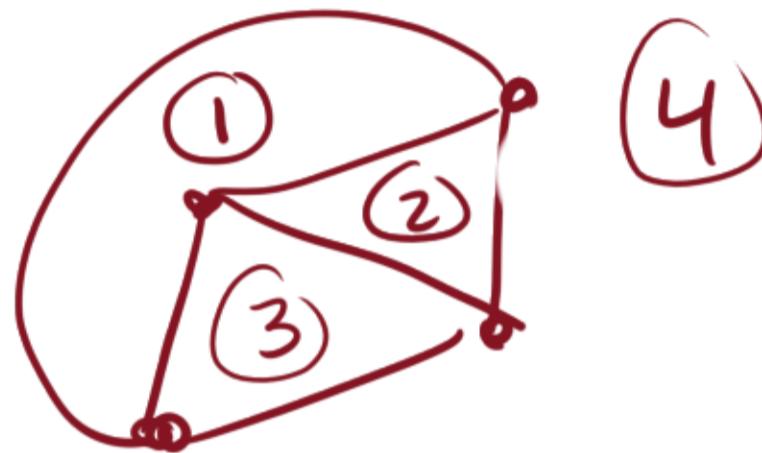
is not.

Polyhedra  $\Rightarrow$  <sup>connected</sup> planar graphs



Def' A face of a planar graph<sup>(multi)</sup> is a region circumscribed by a cycle of edges.  
 (It can include  $\infty$ .) = pieces left if you cut along the graph

Eg.

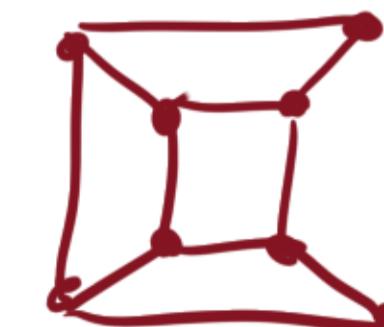
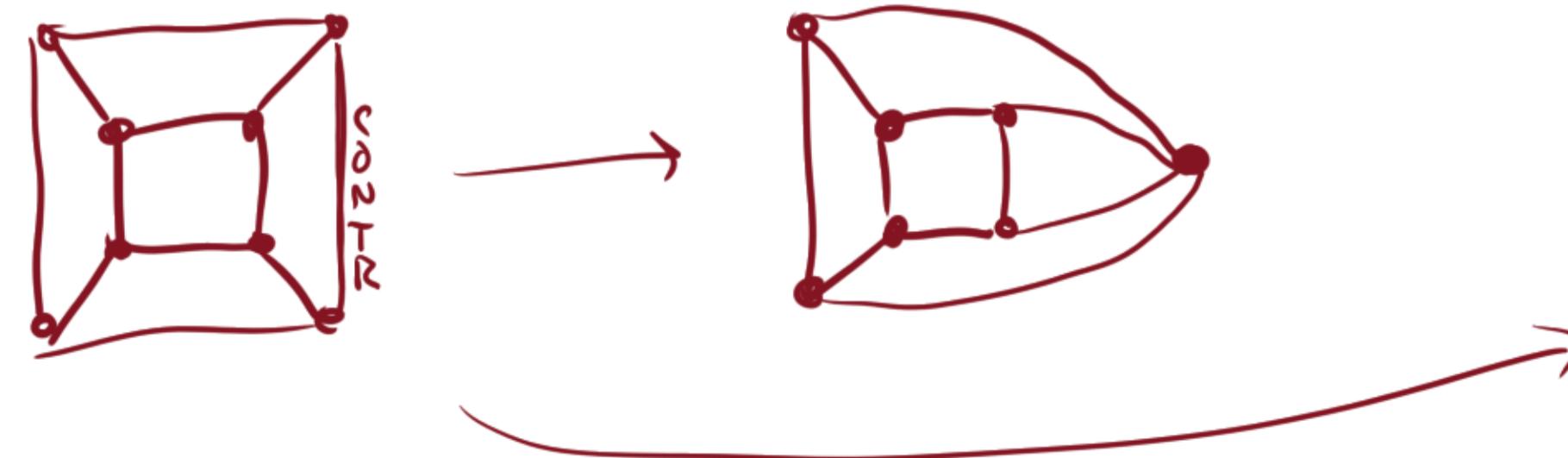


6 edges  
4 faces  
4 vertices

$$4 - 6 + 4 = 2. \checkmark$$

Theorem. Any connected planar graph<sup>multi</sup> with  $E$  edges,  $V$  vertices and  $F$  faces satisfies  $V - E + F = 2$ .

Induct on # vertices!



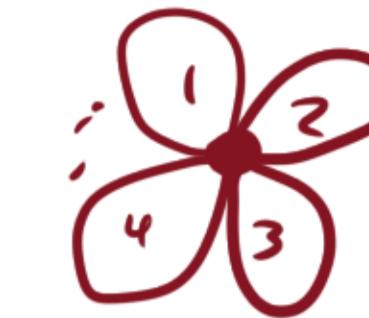
Proof. We will induct on # of vertices.

Base Case: A planar connected multigraph on 1 vertex has n loops:

1 vertex  
n edges  
 $n+1$  faces

And

$$1 - n + (n+1) = 2$$



Inductive Step: Let  $n > 1$ . Suppose any connected planar multigraph with  $n-1$  vertices satisfies  $V-E+F=2$ .

Consider a conn. pl. multigraph w/  $n$  ver!.

Choose an edge connecting 2 vertices (this exists since  $n > 1 \Rightarrow$  graph connected).

Contract it.

The resulting multigraph has 1 fewer vertex and 1 fewer edge.

So  $V-E+F=2$  still holds!

