Theorem. On a tree, there is exactly one path between any 2 vertices.

Pf. **Existence:** Immediate since a tree is connected.

Uniqueness: Assume, for a contradiction, that there are 2 different paths

$$\begin{array}{l} v_1 \sim v_2 \sim \cdots \sim v_n \\ \shortparallel \qquad\qquad\qquad\quad \shortparallel \\ w_1 \sim w_2 \sim \cdots \sim w_m \end{array} \qquad \text{where } v_1 = w_1 \text{ and } v_n = w_m.$$

Let $v_k \neq w_k$ be the 1st position where the paths differ.

Then $v_{k-1} = w_{k-1}$ so

$$\begin{array}{l} v_{k-1} \sim v_k \sim \cdots \sim v_n \\ \shortparallel \qquad\qquad\qquad\quad \shortparallel \\ w_{k-1} \sim w_k \sim \cdots \sim w_m \end{array} \qquad \text{are 2 different paths from } v_{k-1} \text{ to } v_n.$$

So without loss of generality, assume $v_2 \neq w_2$.

Consider $v_2 \sim v_3 \sim \cdots \sim v_n$.

Let w_ℓ be the 1st vertex on $w_2 \sim w_3 \sim \cdots \sim w_m$ which appears anywhere in the $v_2 \sim v_3 \sim \cdots \sim v_n$ path.

Suppose $w_\ell = v_k$.

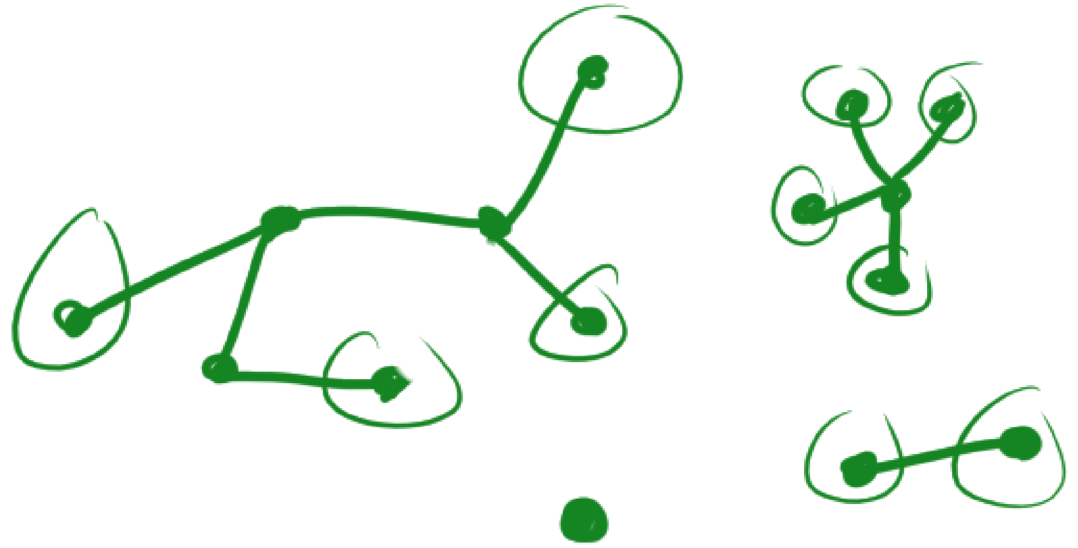
Then we have

$$v_1 \sim v_2 \sim v_3 \sim \cdots \sim w_\ell$$
$$v_1 \sim w_2 \sim w_3 \cdots \sim w_\ell$$

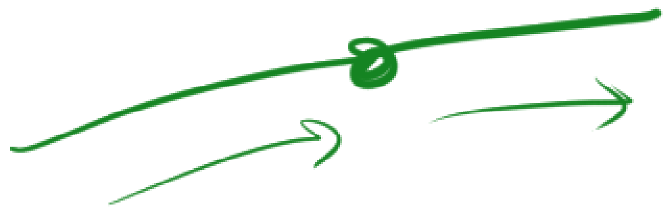
Note:
① no v's coincide since v's form a path
② no w's " " w's " "
③ no v's coincide with w's by construction

So this is a cycle. But trees don't have cycles. $\rightarrow\leftarrow$ $\square$

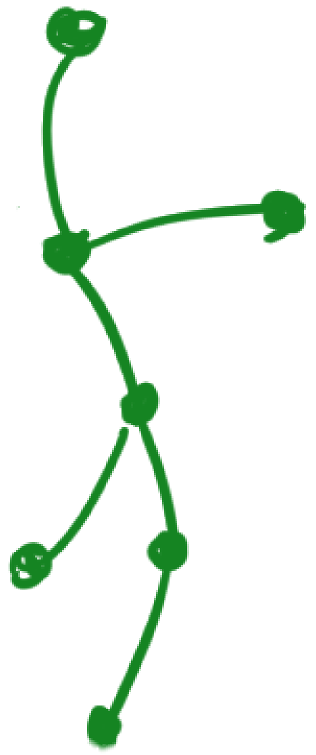
<u>Def^n</u>. A <u>leaf</u> is a vertex of degree 1.



<u>Theorem</u>. Every tree with at least 2 vertices has a leaf.



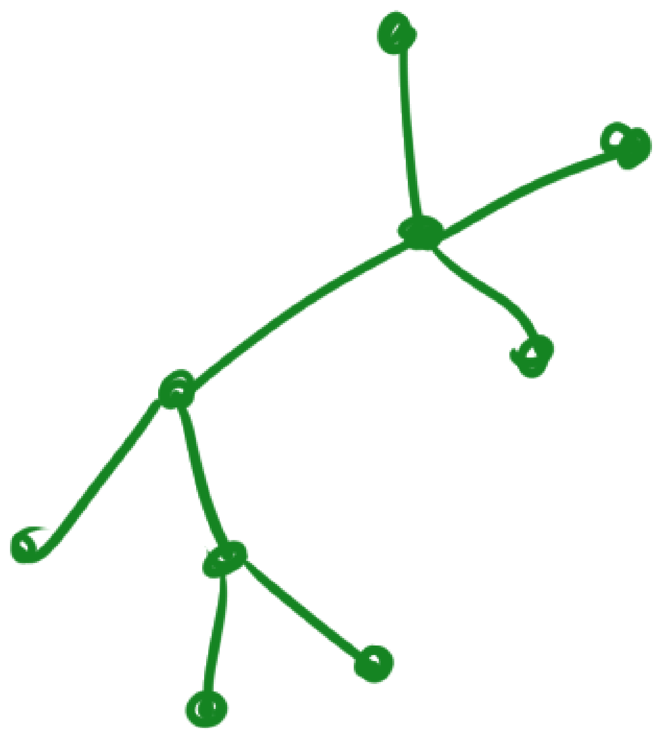
<u>Theorem</u>. Let T be a tree, and v a leaf on T.
Then $T - v$, the tree with that leaf and its edge deleted, is still a tree.



Theorem. Let T be a tree with $n \geq 1$ vertices. Then T has $n-1$ edges.

"There's one fewer edge than there are vertices"

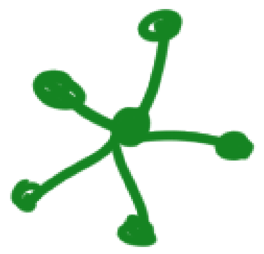
Example.



$|V| = 9$

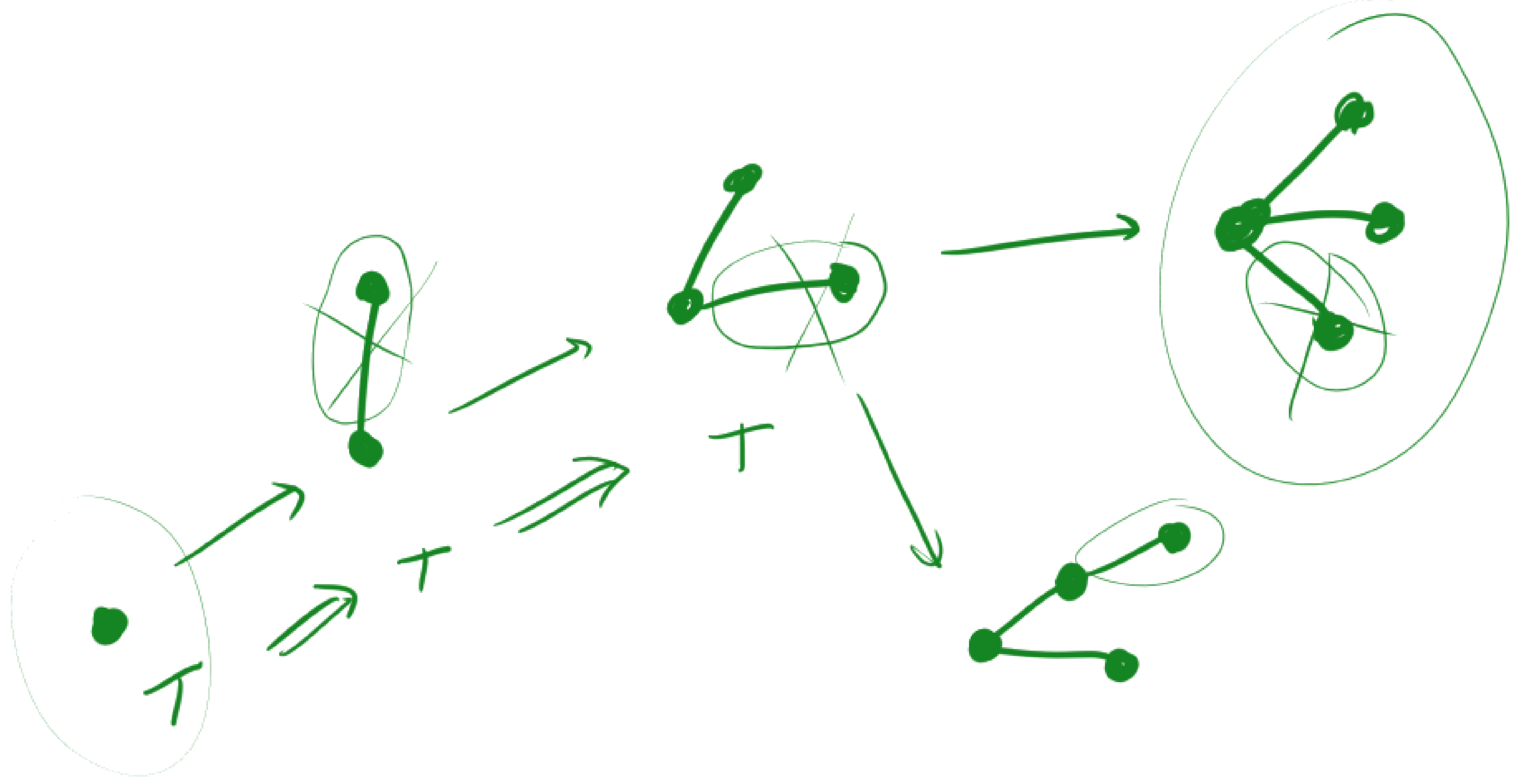
$|E| = 8$

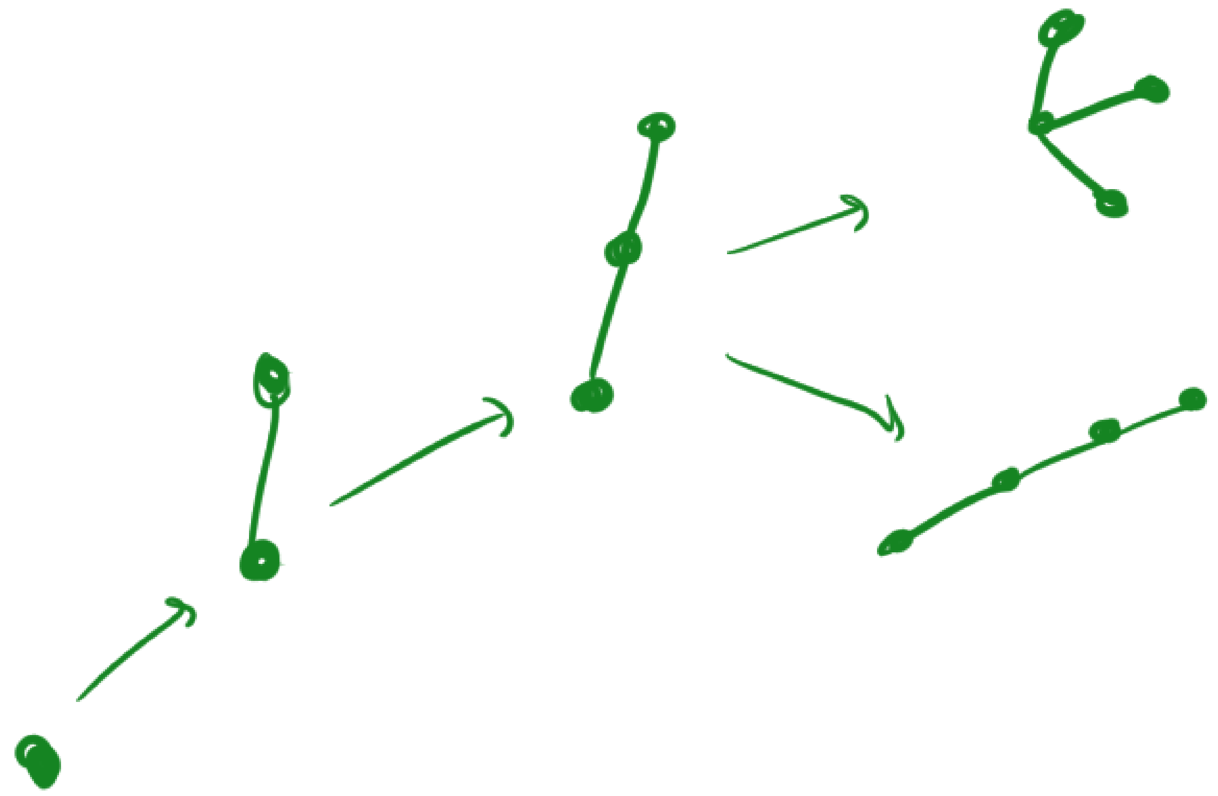
$|V| = 2$
$|E| = 1$



$|V| = 6$
$|E| = 5$

$|V| = 5$
$|E| = 4$





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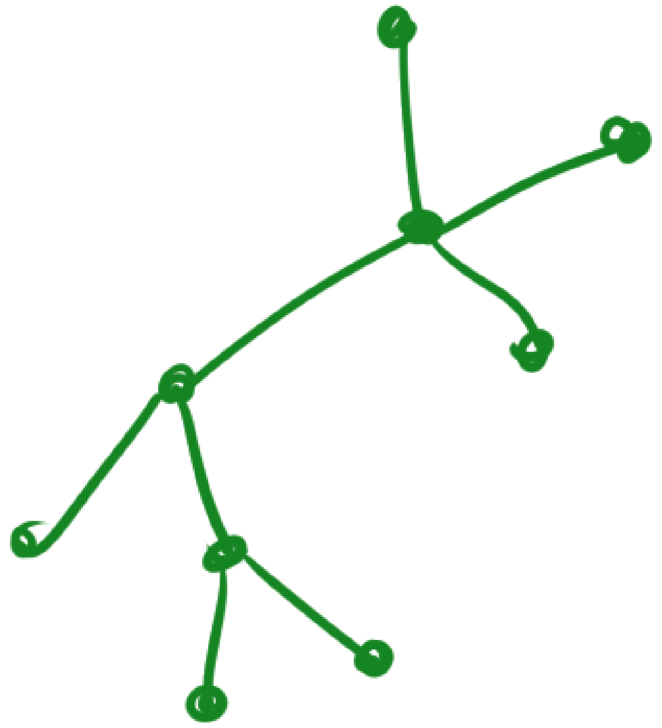

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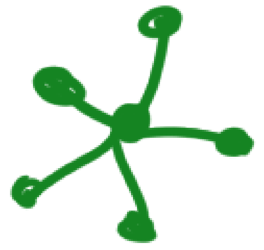

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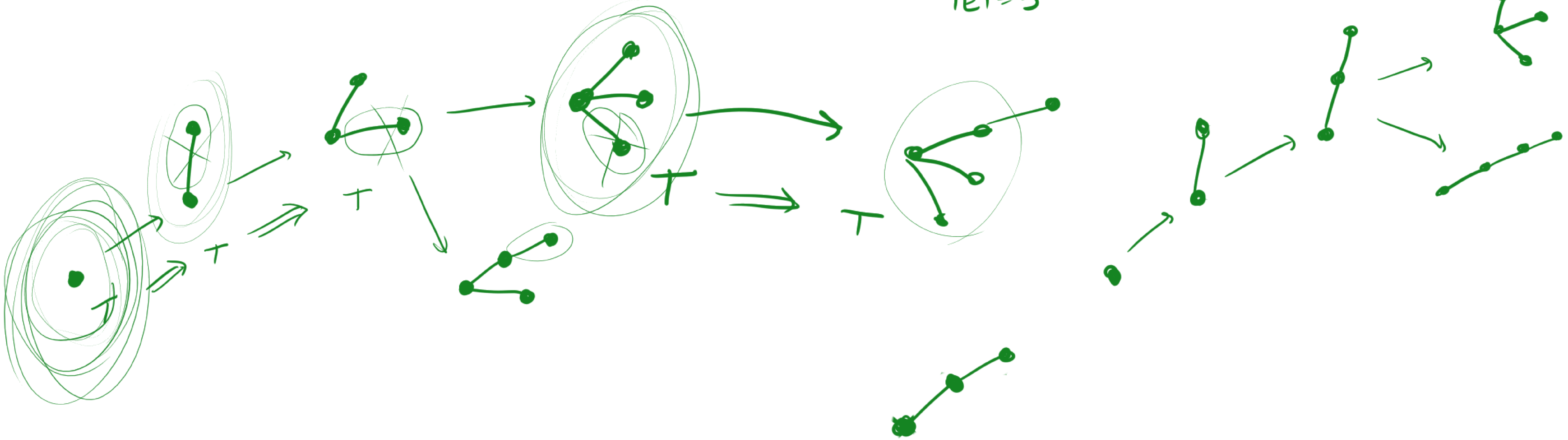

Proof idea:

1. Every tree can be built out of smaller trees.
2. The truth of the statement is preserved by the building process.
3. The statement holds for the smallest trees.

Deeper:

1. If T is a tree (with $n \geq 2$ vertices), it has a leaf, say v.

   $T - v$ is still a tree.

   $T - v$ has one fewer edge
   and one fewer vertex.

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   $|V| = 1$

   $|E| = 0$ ✓

2. From $T - v$ to T, $\begin{cases} \text{vertices increase by } 1 \\ \text{edges increase by } 1. \end{cases}$

   So if $|V| = |E| + 1$ for $T - v$

   then $|V| = |E| + 1$ for T

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   $T-v$ has one fewer edge and one fewer vertex.

3. $|V| = 1$, $|E| = 0$ ✓

2. From $T-v$ to T, vertices increase by 1, edges increase by 1.

   So if $|V| = |E| + 1$ for $T-v$
   then $|V| = |E| + 1$ for T

Standard Format

Base Case: Show this is true for $n=1$ or an appropriate list of small n.

Inductive Step: Show: If the theorem is true for cases smaller than n, then it's true for n.

Proof. Base Case: There is only one tree with one vertex, which has no edges (•).
Therefore the tree w/ 1 vertex has 0 edges. (ie. thm holds for ↑)

Inductive Step: Suppose $n > 1$.

Suppose all trees with k vertices, where $k < n$, have $k-1$ edges. } Inductive Hypothesis

Let T be any tree with n vertices.

Then T has a leaf, say v.

Then $T - v$ is a tree with $n-1$ edges.

Since $n-1 < n$, by the inductive hypothesis, it has $n-2$ edges.

But then T has $n-1$ edges.

Therefore any tree with n vertices has $n-1$ edges. ∎