Equivalence Relations create Equivalence Classes

Ex. "having the same cardinality" on sets

sets of size 1: $\{\bullet\}$ $\{1\}$ $\{3\}$

sets of size 2: $\{1,2\}$ $\{\bullet, \triangle\}$

sets of size 3: . . .

. . .

sets of size $|\mathbb{N}|$: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$

Ex. "Having the same parity" on $\mathbb{Z}$

even: $0, 2, 4, \bullet, 6, -2, \ldots$

odd: $1, 3, \bullet, -17, 13, \ldots$

$$\text{odd} = [1] = [3] = [-17]$$

Ex. "Having the same sign" on $\mathbb{R}$

zero: 0

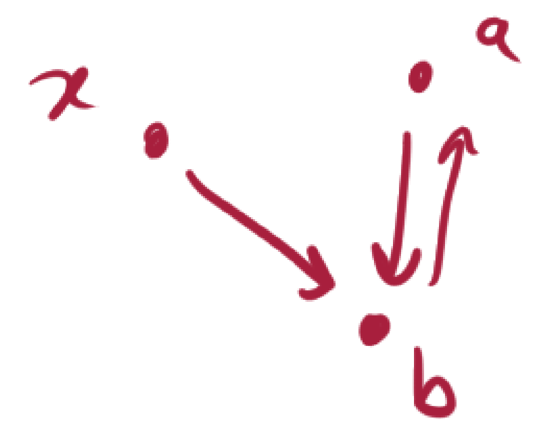
+ve: $\pi, 3, 17, e$

-ve: $-\sqrt{2}, -7, \ldots$

Defⁿ. Let R be an equivalence relation on a set A.
Given $a \in A$, the equivalence class containing a
is $$[a] = \{x \in A : xRa\}.$$
i.e. all elements that relate to a.

Equivalence Relation:

1. Reflexive
2. Symmetric
3. Transitive.



Theorem. Suppose R is an equivalence relation on a set A.
Suppose $a, b \in A$. Then $[a] = [b]$ if and only if aRb.

Pf. Suppose $[a] = [b]$.
First, $b \in [b]$ since bRb since R is reflexive.
Therefore $b \in [a]$ since $[b] = [a]$.
So bRa.

Therefore aRb, by symmetry of R.

Suppose aRb.

First we show $[a] \subseteq [b]$
Let $x \in [a]$.
Then xRa.
Since aRb, by transitivity,
xRb.
Then $x \in [b]$.

Then, we show $[b] \subseteq [a]$.
Let $x \in [b]$.
Then xRb. By symmetry, since aRb, we have bRa.
By transitivity, xRa.

Then $x \in [a]$.

So $[a] = [b]$

Def^n A **partition** of a set A is a set of non-empty subsets of A, say $B_1, B_2, \ldots \subseteq A$ such that $\bigcup B_i = A$ (their union is A) and $B_i \cap B_j = \emptyset$ for $i \neq j$.

Theorem. Suppose R is an equivalence relation on a set A. Then the set $\{[a] : a \in A\}$ of equivalence classes of R forms a partition of A.

Pf. First, we'll show $[a] \subseteq A$ is nonempty.
By definition, $[a] \subseteq A$.
By reflexivity, aRa, so $a \in [a]$.
Therefore $[a] \neq \emptyset$.

Second, we'll show their union is all of A.

Let $x \in \bigcup_{a \in A} [a]$.
Therefore $x \in [a]$ for some a.
Since $[a] \subseteq A$,
then $x \in A$.

Let $x \in A$.
Since xRx by reflexivity,
$x \in [x]$.
Then $x \in \bigcup_{a \in A} [a]$

Third, we'll show $[a] \cap [b] = \emptyset$ unless $[a] = [b]$.

Assume $[a] \cap [b] \neq \emptyset$.
Then there exists $x \in [a] \cap [b]$.
Then xRa and xRb.
Then aRx and xRb, by symmetry.
By transitivity,
then aRb.
Then $[a] = [b]$, by the last theorem.

Theorem. Let $B_1, B_2, \ldots$ be a partition of A.
Then there exists an equivalence relation R on A whose equivalence classes are exactly $B_1, B_2, \ldots$

Pf. We define R to be

$$x R y \text{ if } \exists\ B_i \text{ with } x \in B_i \text{ and } y \in B_i.$$

We must show R is an equivalence relation.

① Reflexivity: Let $x \in A$. Then $x \in B_i$ for some i, since $A = \bigcup B_i$. So $x R x$.

② Symmetry: Suppose $x R y$. Then $\exists\ B_i$ with $x \in B_i$ and $y \in B_i$. Then $y R x$.

③ Transitivity: Suppose $x R y$ and $y R z$.

Then $\exists\ B_i$ with $x, y \in B_i$.
And $\exists\ B_j$ with $y, z \in B_j$.
Since $y \in B_i \cap B_j$, we have $B_i = B_j$ (by def^n of a partition).
So $x, y \in B_i$ and $y, z \in B_i$. So $x, z \in B_i$.
So $x R z$.

We need also show that the B_i's are the equivalence classes.

Let $a \in A$.

Let $a \in B_i$.

We'll show $[a] = B_i$.

$\vdots$