Defn Let A and B be sets.

1. $|A| = |B|$ if $\exists\ f: A \to B$ bijective.
2. $|A| \leq |B|$ if $\exists\ f: A \to B$ injective.
3. $|A| < |B|$ if $\exists\ f: A \to B$ injective but $\nexists\ f: A \to B$ bijective.

$$\begin{tikzcd} A \arrow[r, shift left, "\text{inj}"] & B \arrow[l, shift left, "\text{inj}"] \end{tikzcd}$$
$$\Downarrow$$
$$\begin{tikzcd} A \arrow[r, leftrightarrow, "\text{bij}"] & B \end{tikzcd}$$

We saw before that $|\mathbb{N}| \neq |\mathbb{R}|$.

But $f: \mathbb{N} \to \mathbb{R}$ given by $f(n) = n$ is injective.

So $|\mathbb{N}| < |\mathbb{R}|$.

Defn Write $\aleph_0$ for $|\mathbb{N}|$ (aleph-nought)

So $\aleph_0 < |\mathbb{R}|$.

Theorem. Let A be any set. Then $|A| < |\mathcal{P}(A)|$.

Pf. First, we find an injection $f: A \to \mathcal{P}(A)$

Define $f(x) = \{x\}$.

If $f(x_1) = f(x_2)$

then $\{x_1\} = \{x_2\}$, so $x_1 = x_2$.

So f is injective.

Next, we will show there is no bijection $f: A \to \mathcal{P}(A)$.

Let $f: A \to \mathcal{P}(A)$. We will show it is not surjective,

by finding $B \in \mathcal{P}(A)$ s.t. $B \neq f(a)$ for any $a \in A$.

Define $B = \{a \in A : a \notin f(a)\} \in \mathcal{P}(A)$.

Let $a \in A$. We'll show $f(a) \neq B$.

Either $a \in f(a)$ or $a \notin f(a)$.

Case I:

If $a \notin f(a)$

then $a \in B$.

So $B \neq f(a)$.

Case II:

If $a \in f(a)$

then $a \notin B$.

So $B \neq f(a)$.

$\blacksquare$

The Cantor - Bernstein - Schröder Theorem

If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Translation: If there are injections $f: A \to B$ and $g: B \to A$ then there's a bijection $A \to B$.

Example Application:

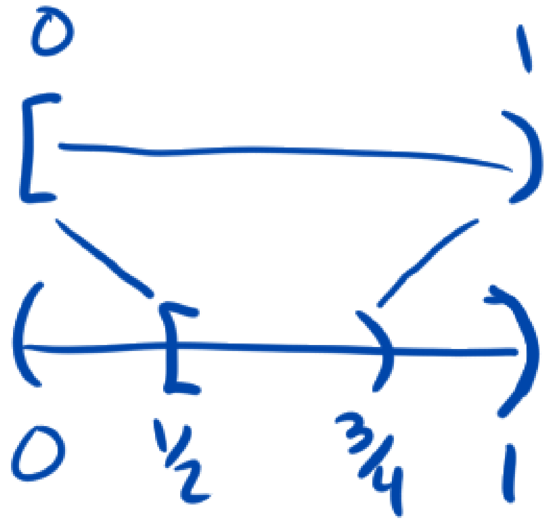
$$|[0,1)| = |(0,1)|$$

Proof 1: Come up with a bijection. (?)

Proof 2: Use CBS Theorem: Come up with injections both ways:

$f: (0,1) \longrightarrow [0,1)$, $f(x) = x$.

$g: [0,1) \longrightarrow (0,1)$, $f(x) = \frac{1}{2}x + \frac{1}{4}$



The Cantor-Bernstein-Schröder theorem

If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.

Translation: If there are injections $f: A \to B$ and $g: B \to A$ then there's a bijection $A \to B$.

Example Application:

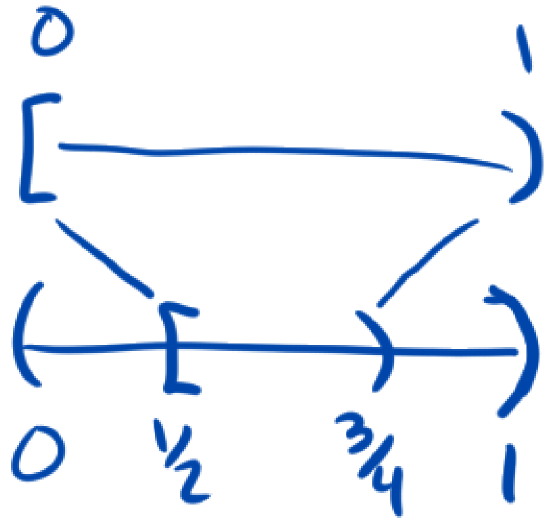
$$|[0,1)| = |(0,1)|$$

Proof 1: Come up with a bijection. (?)

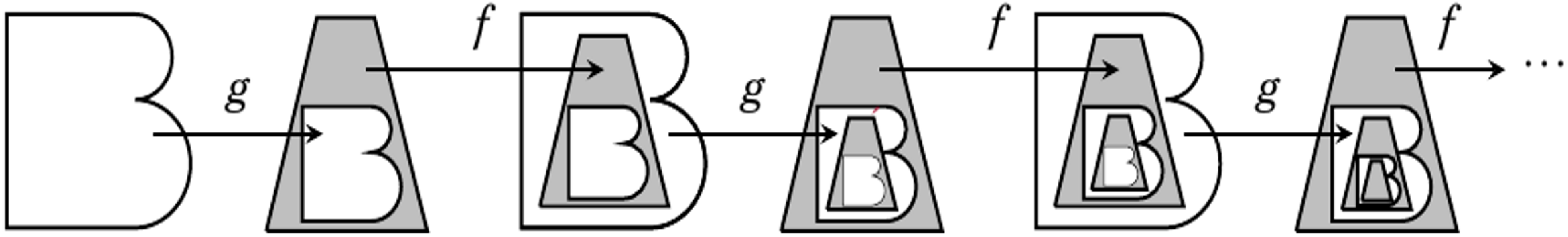
Proof 2: Use CBS Theorem: Come up with injections both ways:

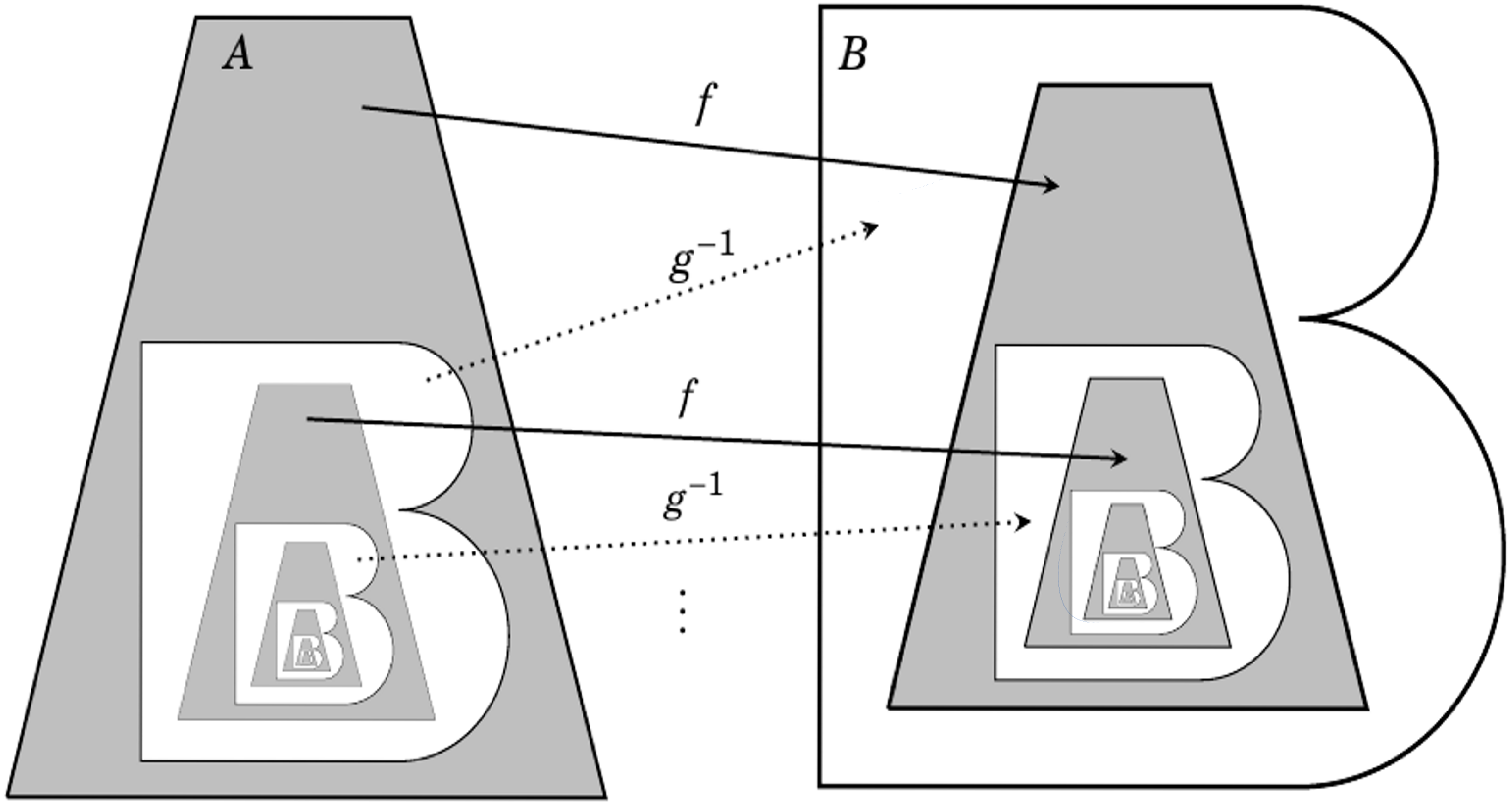
$f: (0,1) \longrightarrow [0,1)$, $f(x) = x$.

$g: [0,1) \longrightarrow (0,1)$, $f(x) = \frac{1}{2}x + \frac{1}{4}$



$f: A \to B$ injection and $g: B \to A$ injection

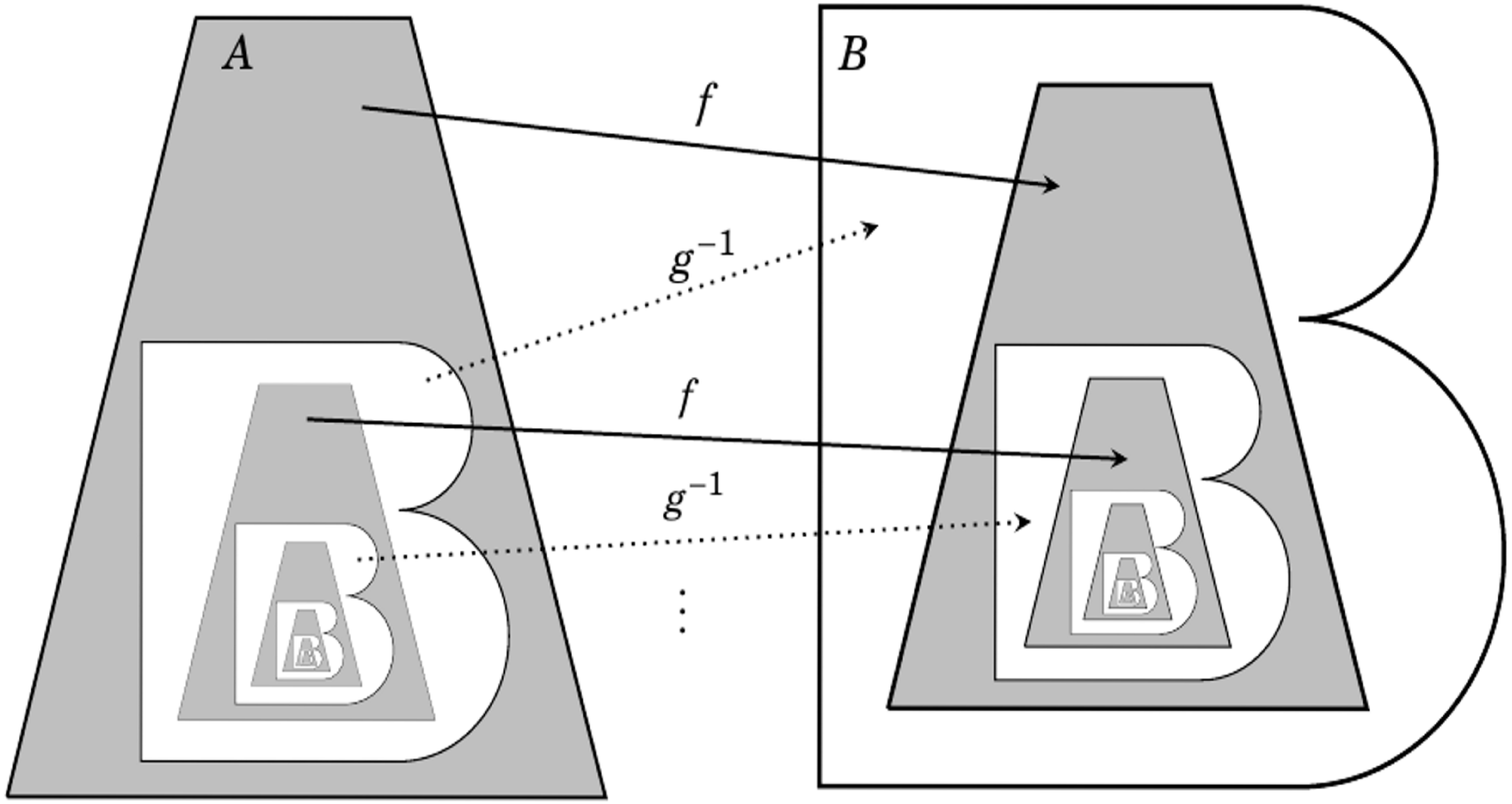




$$\text{Grey} \xrightarrow[\text{bijective}]{f} \text{Grey}$$

$$\text{White} \xrightarrow[\text{bijective}]{g^{-1}} \text{White}$$

Define $h: A \longrightarrow B$



$$\text{Grey} \xrightarrow{f} \text{Grey}$$
bijective

$$\text{White} \xrightarrow{g^{-1}} \text{White}$$
bijective

Define $h: A \longrightarrow B$

$$h(x) = \begin{cases} f(x) & \text{if } x \in \text{grey part} \\ g^{-1}(x) & \text{if } x \in \text{white part} \end{cases}$$

Question: Does there exist a set A s.t. $|\mathbb{N}| < |A| < |\mathbb{R}|$?

Continuum hypothesis: no (1880's)

Is it true?

In the standard axioms of set theory: **not decidable**.