Theorem. Let $f: A \to B$. Then f is bijective if and only if f has an inverse. "iff"

$$\boxed{\begin{tikzcd} A \arrow[r, shift left, "f"] & B \arrow[l, shift left, "f^{-1}"] \end{tikzcd}}$$
$a \qquad b$

If f is bijective, then it has an inverse
and
If f has an inverse, then it is bijective

P "if and only if" Q
$P \iff Q$

Proof. ($\Rightarrow$) Assume f is bijective.
We will define $f^{-1}: B \to A$ as follows.
Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$.
So let $f^{-1}(b) = a$. Since, by injectivity of f, there is only one such a, this is well-defined.

Claim: $f^{-1} \circ f = I_A$
Let $a \in A$. Let $b = f(a)$. Then $f^{-1}(b) = a$.
So $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$.

Claim: $f \circ f^{-1} = I_B$
Let $b \in B$. Let $a = f^{-1}(b)$. Then $f(a) = b$.
Then $f \circ f^{-1}(b) = f(f^{-1}(b)) = f(a) = b$.

Therefore f has an inverse.

($\Leftarrow$) Assume f has an inverse, namely $f^{-1}: B \to A$.
First, we will show f is surjective.
Suppose $b \in B$. Let $a = f^{-1}(b)$. (using f^{-1} is inverse)
Then $f(a) = f(f^{-1}(b)) = f \circ f^{-1}(b) = I_B(b) = b$.

Next, we will show f is injective.
Let $a_1, a_2 \in A$. Suppose $f(a_1) = f(a_2)$. in B

$$f^{-1}(f(a_1)) = f^{-1}(f(a_2))$$
$$\underbrace{f^{-1} \circ f}(a_1) = \underbrace{f^{-1} \circ f}(a_2)$$
$$I_A(a_1) = I_A(a_2)$$

Then $a_1 = a_2$.
Therefore f is bijective.

Cardinalities

Def^n. A set A has cardinality $n \in \mathbb{Z}$ if there's a bijection $f: \{1, 2, \ldots, n\} \to A$.

x	$f(x)$
1	1st ladybug
2	2nd ladybug
$\vdots$	
n	nth ladybug

(finite or infinite)

Def^n. Two sets A and B have the same cardinality (denoted by $\underline{|A| = |B|}$) if there's a bijection $f: A \to B$.

"$|A|$" is not defined.

Q: Is this a good definition?

Good Properties (for a notion of "sameness"):

1. $|A| = |A|$ ("reflexivity")
2. If $|A| = |B|$ then $|B| = |A|$. ("symmetry")
3. If $|A| = |B|$ and $|B| = |C|$ then $|A| = |C|$. ("transitivity")

Thm. For any set A, $|A| = |A|$.

Pf. Let $f : A \to A$ be the identity function,

$$f = 1_A .$$

This function f is surjective, since for any $a \in A$, $f(a) = a$.

(codom.) domain

This function f is injective, since if $f(a) = f(a')$ then $a = a'$. $\blacksquare$

Scratch:

Task: find a bijection

$$f : A \to A .$$

Cardinalities.

$f: \mathbb{Z} \to 2\mathbb{Z}$ So $|\mathbb{Z}| = |2\mathbb{Z}|$

$f(x) = 2x$

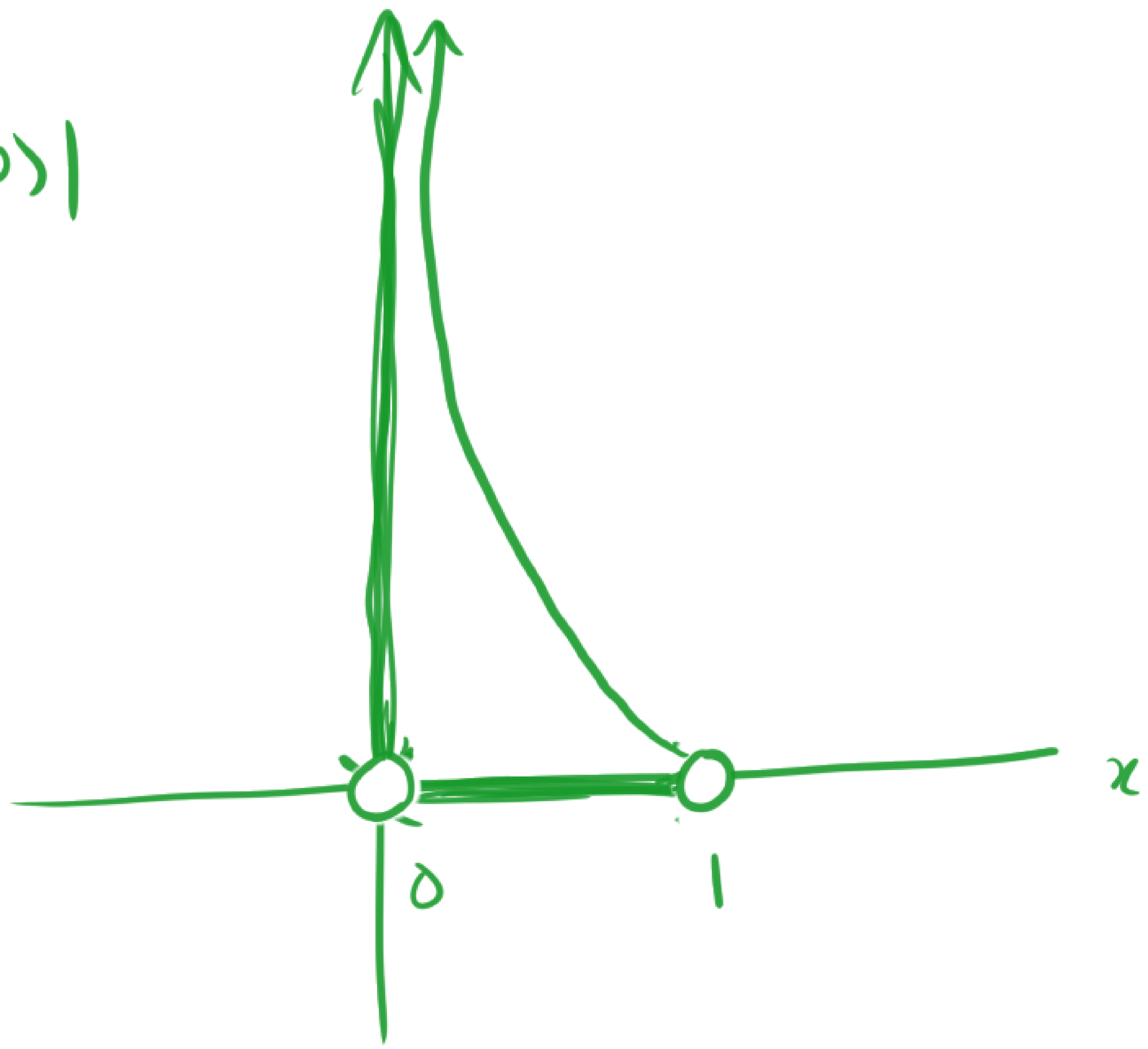
$\mathbb{N}$, $\mathbb{Z}$, $2\mathbb{Z}$, $\mathbb{Z} \times \mathbb{Z}$,

even integers

$\mathbb{Q}$, $\mathbb{Q} \cap [0,1]$, $\mathbb{R}$, $(0,\infty)$,

$(0,1)$, $\mathcal{P}(\mathbb{N})$, $\mathcal{P}(\mathbb{Q})$, $\mathcal{P}(\mathbb{R})$.

$|(0,1)| = |(0,\infty)|$



$\boxed{f: (0,1) \to (0,\infty)}$

$f(x) = \frac{1}{x} - 1$