2. For each of the following functions, determine if it is injective, surjective or bijective.

(a) $f : \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 7x$.

- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(b) $g : \mathbb{R} \to [-1, 1]$ given by $f(x) = \sin(x)$.

- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(c) $h : \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ given by $f((x, y)) = x + y$.

- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

$\mathscr{P}(X) = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}, X \}$

Ex. $k(\{a\}) = X - \{a\} = \{b,c\}$.

$k(\emptyset) = X - \emptyset = X$

$k(\{b\}) = \{a,c\}$

8 elements

8 elements

(d) Let $X = \{a, b, c\}$. Let $k : \mathscr{P}(X) \to \mathscr{P}(X)$ given by $k(Y) = X - Y$ (that's a 'set minus' symbol).

- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

(e) $\ell : \{a, b, c\} \to \{w, x, y, z\}$ given by $f(a) = w$, $f(b) = z$ and $f(c) = y$.

- Injective? YES / NO
- Surjective? YES / NO
- Bijective? YES / NO

- Surjective? YES / NO
- Bijective? YES / NO

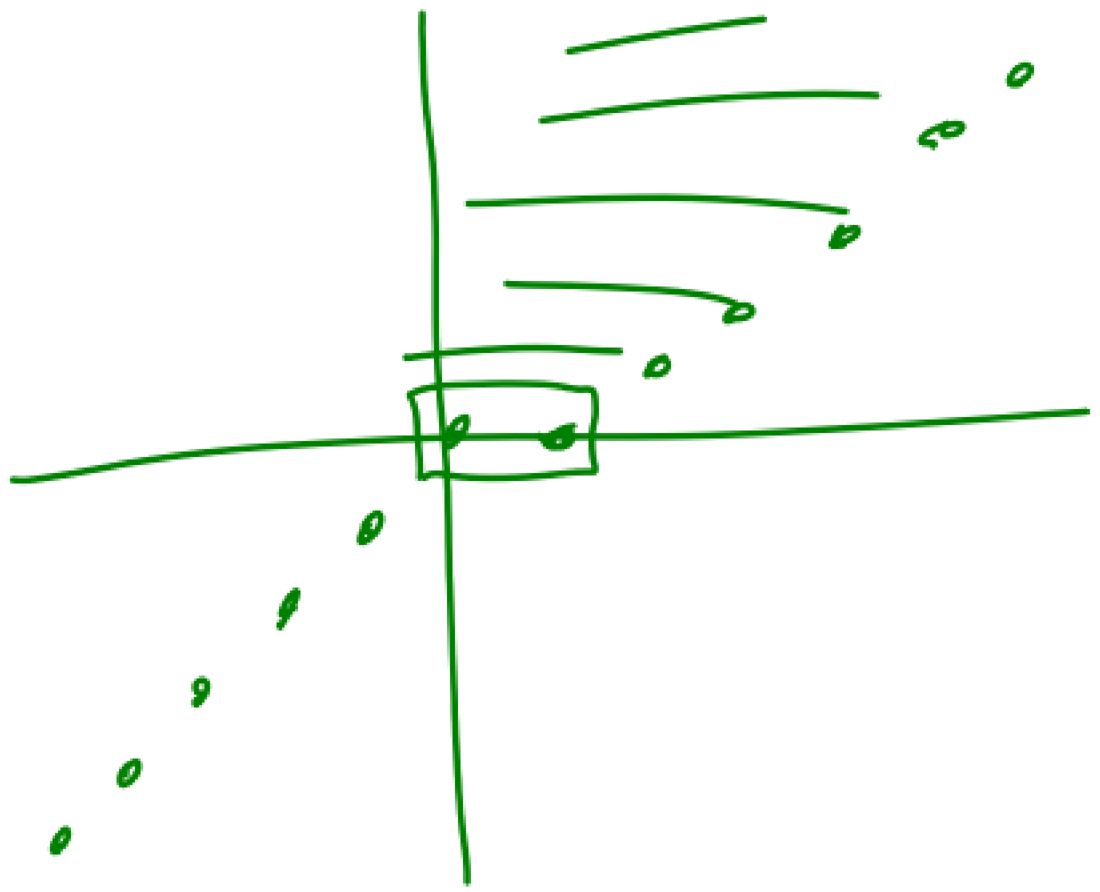
3. If a function f is injective, does that imply f is bijective?

$$YES \quad NO$$

4. Give an example of a function $\boxed{f : \mathbb{Z} \to \mathbb{Z}}$ which is surjective but not injective.

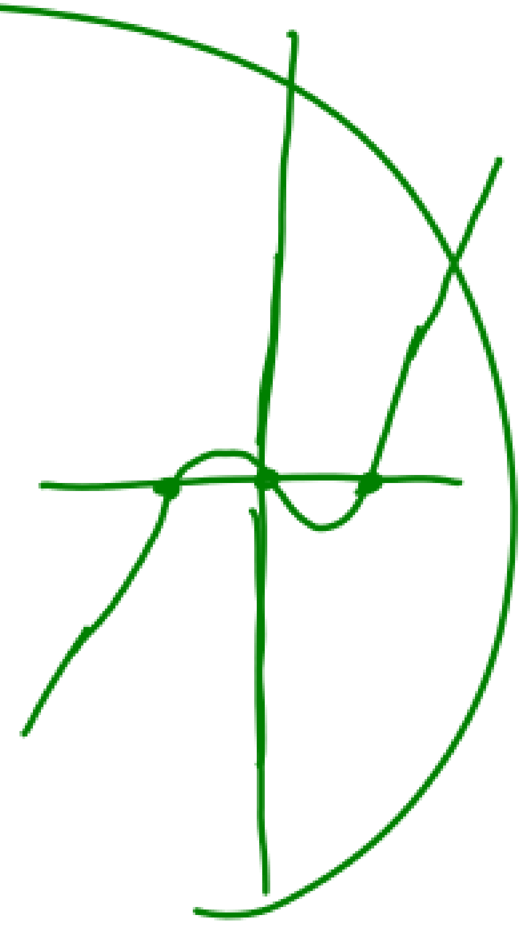
$f : \mathbb{Z} \to \mathbb{Z}$

$$f(x) = \begin{cases} 0 & \text{if } x = 0, 1 \\ x & \text{if } x < 0 \\ x - 1 & \text{if } x > 0 \end{cases}$$



$f : \mathbb{R} \to \mathbb{R}$

$f(x) = x^3 - x$



$x = 2 \quad f(x) = 6 \quad <$

$x = 3 \quad f(x) = 24$

no longer surjective

Defⁿ. Let $f : A \to B$, $g : B \to C$. Then the composition of f and g is the function $g \circ f : A \to C$ given by $g \circ f(a) = g(f(a))$ for each $a \in A$.

Example.

$f : \{1, 2, 3\} \to \{a, b\}$

x	$f(x)$
1	a
2	b
3	b

$g : \{a, b\} \to \{1, 2\}$

x	$g(x)$
a	1
b	2

$h : \{a, b, c\} \to \{1, 2, 3\}$

x	$h(x)$
a	1
b	1
c	3

$f \circ h : \{a, b, c\} \to \{a, b\}$

x	$f \circ h(x)$
a	a
b	a
c	b

$g \circ f : \{1, 2, 3\} \to \{1, 2\}$

x	$g \circ f(x)$
1	1
2	2
3	2

$h \circ f$ not defined.

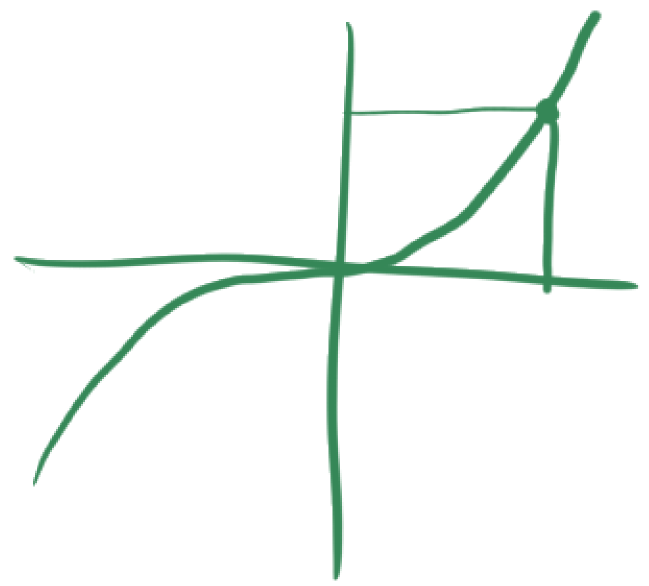
(according to our definition)

Defⁿ. Let A be a set. The identity function $1_A : A \to A$ is given by $f(a) = a$ for all $a \in A$.

Defⁿ. Let $f : A \to B$. A function $g : B \to A$ is the <u>inverse</u> of g if

$$f \circ g = 1_B \quad \text{and} \quad g \circ f = 1_A .$$

In this case, we write $g = f^{-1}$.



Example. $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^3 + 1$.

Define $f^{-1} : \mathbb{R} \to \mathbb{R}$ given by $f^{-1}(x) = \sqrt[3]{x-1}$.

Prove it is the inverse:

$$\begin{aligned} f \circ f^{-1}(x) &= f\left(f^{-1}(x)\right) \\ &= f\left(\sqrt[3]{x-1}\right) \\ &= \left(\sqrt[3]{x-1}\right)^3 + 1 \\ &= x - 1 + 1 \\ &= x \end{aligned}$$

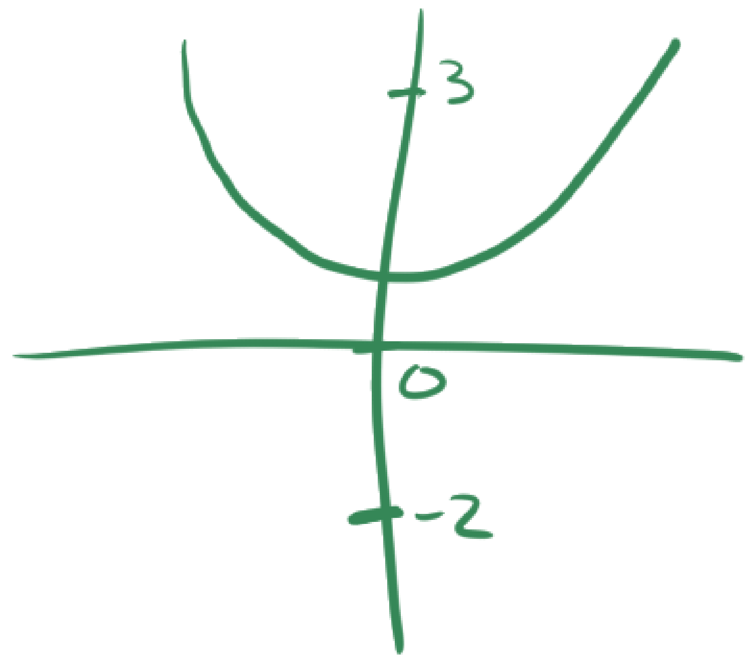
So $f \circ f^{-1} = 1_{\mathbb{R}}$

$$\begin{aligned} f^{-1} \circ f(x) &= f^{-1}(f(x)) \\ &= f^{-1}(x^3 + 1) \\ &= \sqrt[3]{(x^3+1) - 1} \\ &= \sqrt[3]{x^3} \\ &= x \end{aligned}$$

So $f^{-1} \circ f = 1_{\mathbb{R}}$. $\square$

Example.

$f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^2 + 1$.



There is no inverse!

Why not?

What would $f^{-1}(0)$ be? No answer.

" " $f^{-1}(3)$ be? Two answers.

Ex. $g: \{a,b,c\} \to \{1,2,3\}$

x	$g(x)$
a	1
b	1
c	3

1 reason: If g^{-1} existed then

$$g^{-1}(1) = g^{-1}(g(a)) = 1_{\{a,b,c\}}(a) = a$$

and $g^{-1}(1) = g^{-1}(g(b)) = 1_{\{a,b,c\}}(b) = b$.

$\rightarrow\leftarrow$.

Theorem. Let $f: A \to B$. Then $\underbrace{f \text{ is bijective if and only if } f \text{ has an inverse.}}$ "iff"

If f is bijective, then it has an inverse
and
If f has an inverse, then it is bijective

P "if and only if" Q
$P \Longleftrightarrow Q$

Proof. ($\Rightarrow$) Assume f is bijective.
We will define $f^{-1}: B \to A$ as follows.

Claim: $f^{-1} \circ f = I_A$

Claim: $f \circ f^{-1} = I_B$

Therefore f has an inverse.

($\Leftarrow$) Assume f has an inverse.
First, we will show f is surjective.

Next, we will show f is injective.

Therefore f is bijective.

Theorem. Let $f: A \to B$. Then f is bijective if and only if f has an inverse. "iff"

If f is bijective, then it has an inverse
and
If f has an inverse, then it is bijective

P "if and only if" Q
$P \Longleftrightarrow Q$

Proof. ($\Rightarrow$) Assume f is bijective.

We will define $f^{-1}: B \to A$ as follows.
Let $b \in B$. Since f is surjective, there exists $a \in A$ such that $f(a) = b$.
So let $f^{-1}(b) = a$. Since, by injectivity of f, there is only one such a, this is well-defined.

Claim: $f^{-1} \circ f = I_A$
Let $a \in A$. Let $b = f(a)$. Then $f^{-1}(b) = a$.
So $f^{-1} \circ f(a) = f^{-1}(f(a)) = f^{-1}(b) = a$.

Claim: $f \circ f^{-1} = I_B$

Therefore f has an inverse.

($\Leftarrow$) Assume f has an inverse.

First, we will show f is surjective.

Next, we will show f is injective.

Therefore f is bijective.