Defⁿ. A function $f: A \to B$

① is <u>surjective</u> if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. ("onto")

② is <u>injective</u> if for every pair $a_1, a_2 \in A$, if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$ "no collisions" ("1-to-1")

[Equivalently, if for every pair $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$]

③ is <u>bijective</u> if it is both injective and surjective.

Ex. $f: \{a, b, c\} \to \boxed{\{1, 2\}}$

x	$f(x)$
a	1
b	1
c	2

<u>challenge</u>	<u>response</u>	<u>verification</u>
1	a or b	$f(a) = 1$, $f(b) = 1$
2	c	$f(c) = 2$

f <u>is</u> surjective because every output <u>occurs</u> for some input

f is <u>not</u> injective because $f(a) = f(b)$ but $a \neq b$.

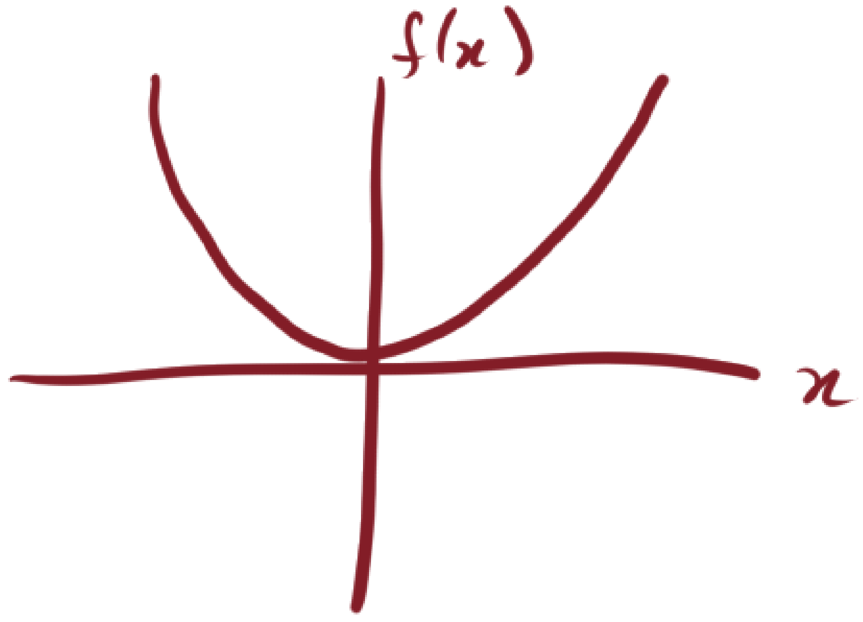
Ex. $g: \{a, b\} \to \{1, 2, 3\}$

x	$f(x)$
a	1
b	2

g is <u>not</u> surjective because 3 does not occur.

g is injective, because there are no collisions.

Example $\quad f: \overset{\text{domain}}{\mathbb{R}} \to \overset{\text{codomain}}{\mathbb{R}}, \quad f(x) = x^2.$



injective? $\boxed{\text{NO}}$

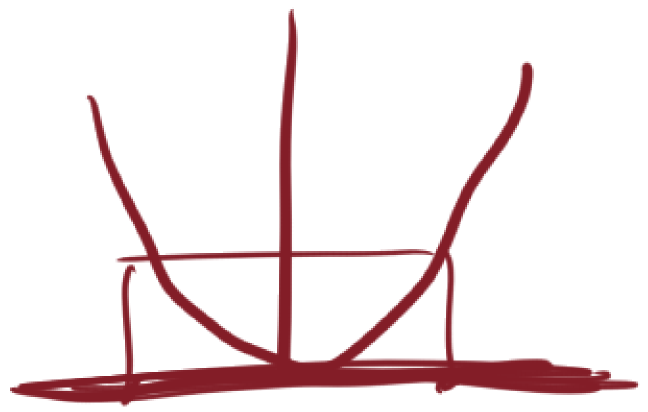
$$\left.\begin{aligned} f(-1) &= (-1)^2 = 1 \\ f(1) &= 1^2 = 1 \end{aligned}\right\} \text{ collision}$$

surjective? $\boxed{\text{NO}}$

There does not exist any $x \in \mathbb{R}$ s.t. $x^2 = -1$.
So -1 does not occur as an output.

bijective? NO

Modification: $\quad f: \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\}, \quad f(x) = x^2$



injective? $\boxed{\text{NO}}$

surjective? $\boxed{\text{YES}}$ (every non-negative real # is a square of a real number)

bijective? $\boxed{\text{NO}}$

$f: A \to B$

If f is a function, then for each input $a \in A$,
there's exactly one output $f(a) \in B$.

f is <u>injective</u> if for each possible output $b \in B$,
there's <u>at most 1</u> input $a \in A$ s.t. $f(a) = b$.

f is <u>surjective</u> if for each possible output $b \in B$,
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Defⁿ The range of a function $f: A \to B$ is $\{ b \in B : \exists\, a \in A,\ f(a) = b \}$.

Ex. $f: \{1,2,3\} \to \{a,b,c\}$

x	$f(x)$
1	a
2	c
3	c

domain $= \{1,2,3\}$

codomain $= \{a,b,c\}$

range $= \{a,c\}$

Theorem. f is surjective if and only if range(f) = codomain(f).

"range of f" "codomain of f".

Example. $f: \mathbb{Z} \to \mathbb{Z}$ given by $f(x) = 2x$.

domain $= \mathbb{Z}$

codomain $= \mathbb{Z}$

range $=$ even integers $= \{2x : x \in \mathbb{Z}\}$

injective? If $\overbrace{2x_1 = 2x_2}^{f(x_1)=f(x_2)}$ then $x_1 = x_2$. YES.

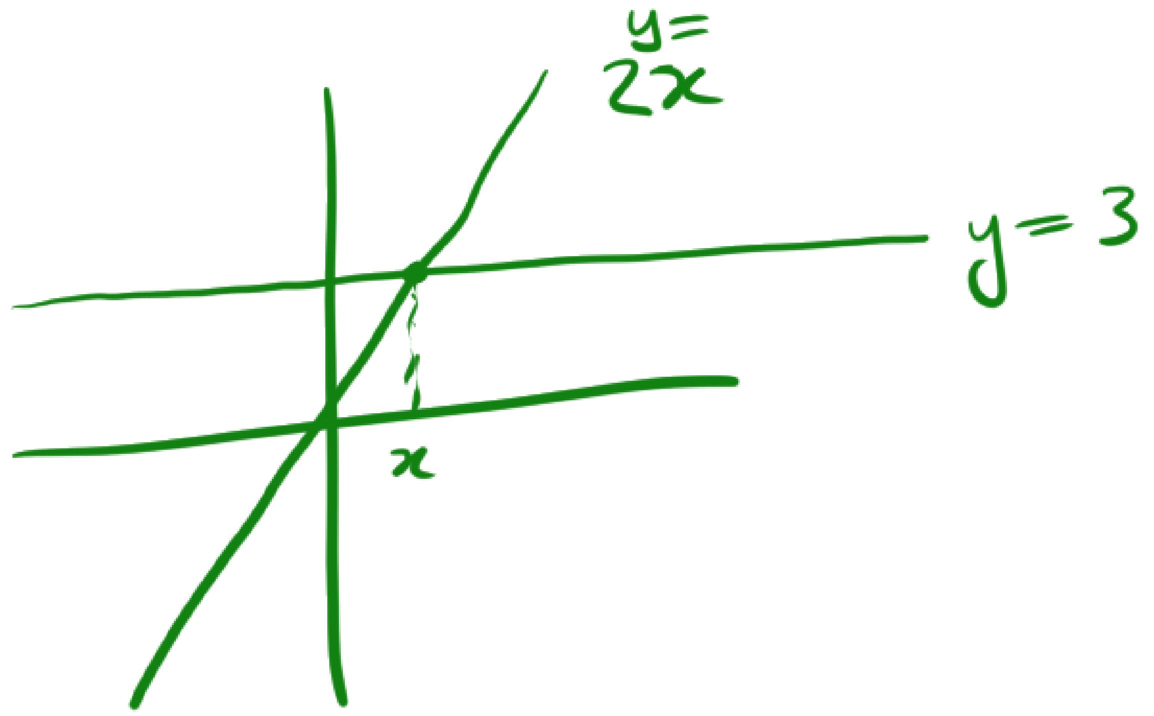
surjective? NO. Reason 1: range $\neq$ codomain

Reason 2: 5 has no corresponding input

($\nexists\, a \in \mathbb{Z}$ s.t. $f(a) = 5$.)

bijective? NO.

Example. $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$.



domain $= \mathbb{R}$

codomain $= \mathbb{R}$

range $= \mathbb{R}$

injective? YES Horizontal line test.

surjective? YES

Given $b \in \mathbb{R}$, $\exists\ a \in \mathbb{R}$, $2a = b$.

(namely, $a = \frac{b}{2}$)

Formal Proof. Let $b \in \mathbb{R}$. (challenge)

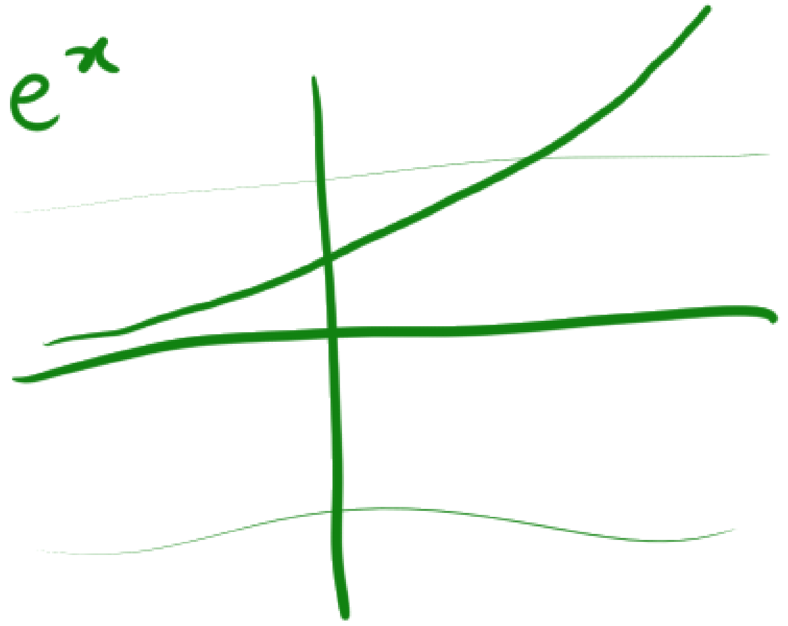
Let $a = b/2$. (response)

Then $a \in \mathbb{R}$ and $f(a) = 2a = 2(b/2) = b$. $\blacksquare$

Give examples of functions $f: \mathbb{R} \to \mathbb{R}$ (e.g. $\sin(x)$, x^2 etc.) which are:

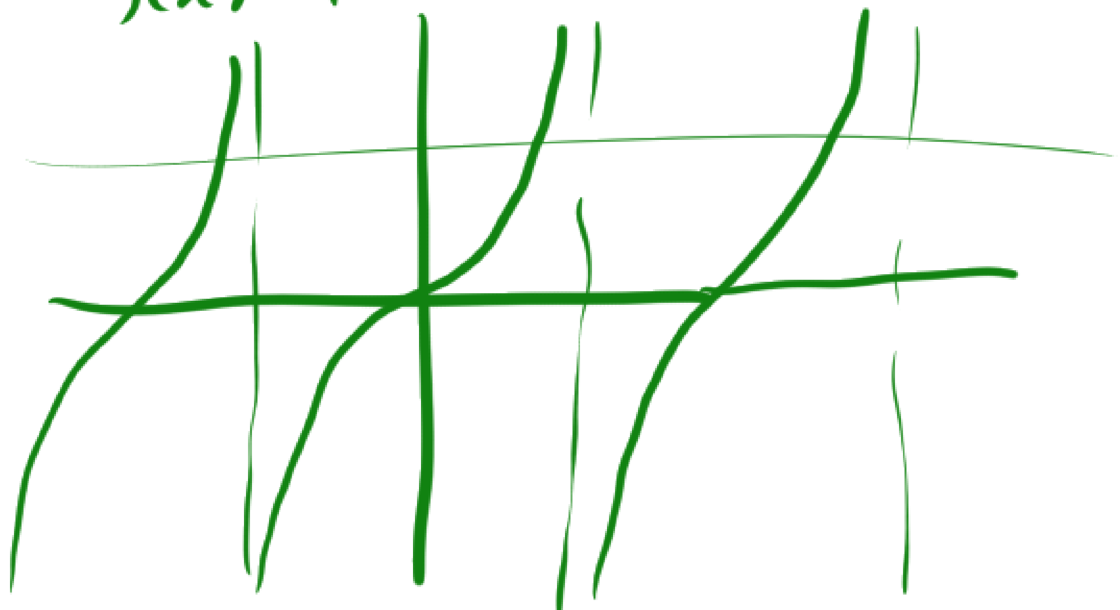
injective but not surjective

e^x

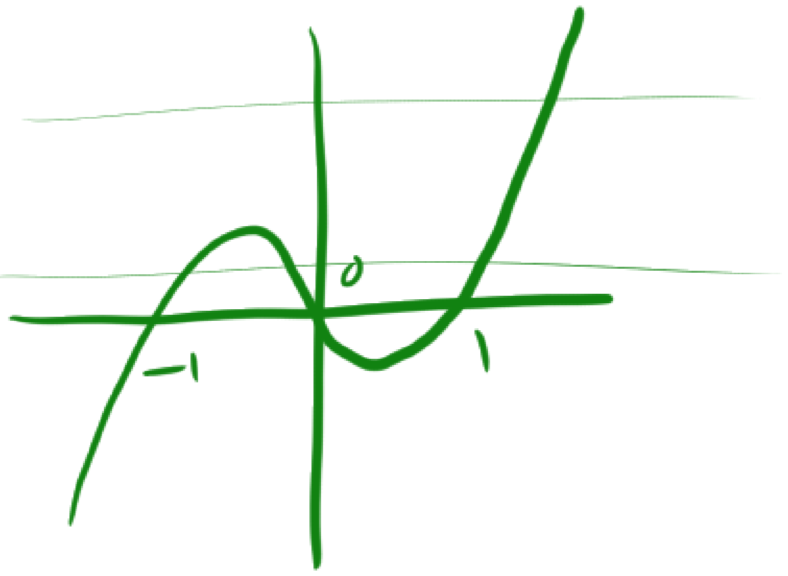


surjective but not injective

$f(x) = \tan x$



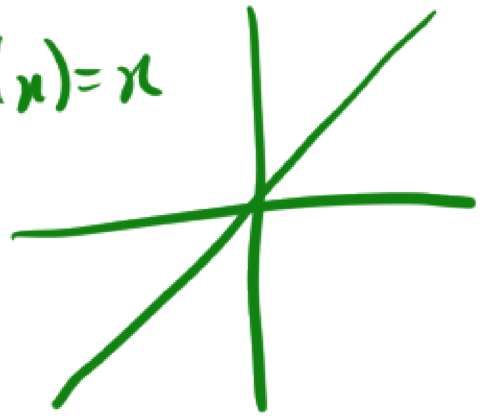
except not defined at some x. Not $\mathbb{R} \to \mathbb{R}$



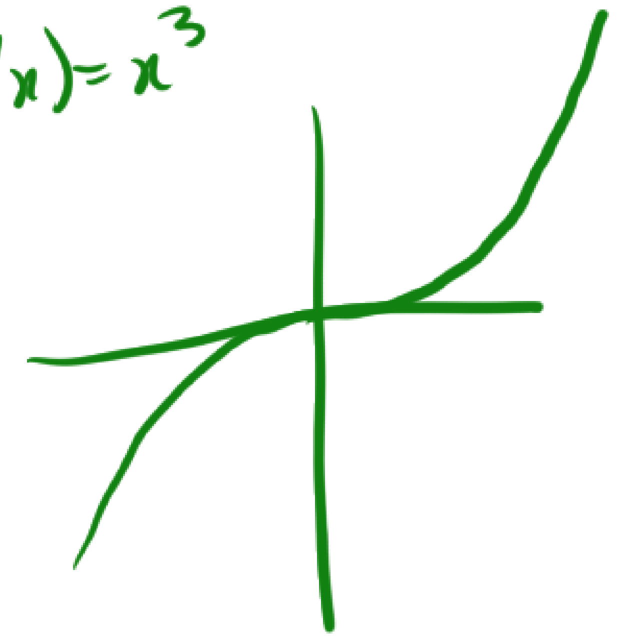
$f(x) = x(x-1)(x+1)$

bijective

$f(x) = x$



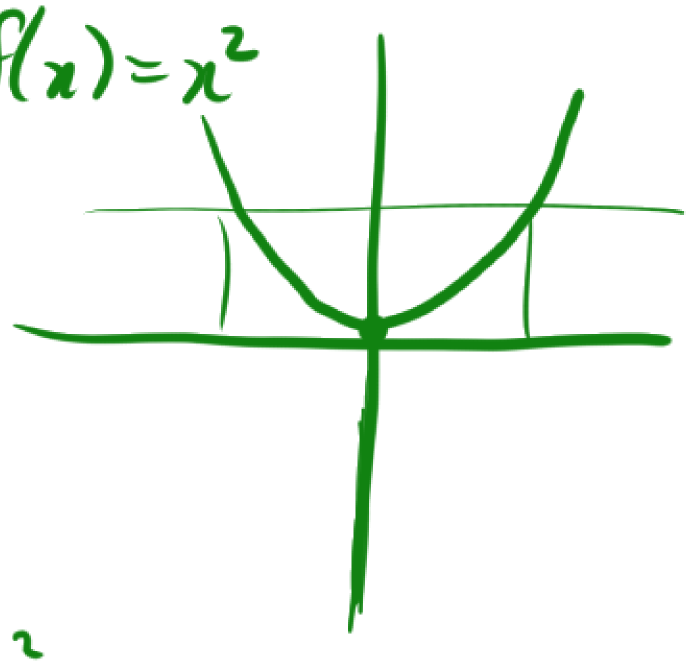
$f(x) = x^3$



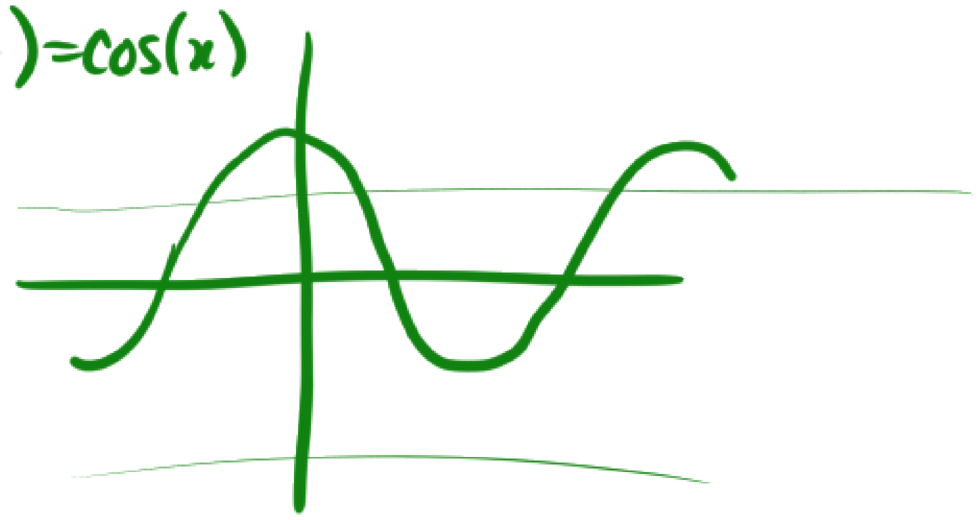
$f(x) = x^5, x^7, \ldots$

neither injective nor surjective

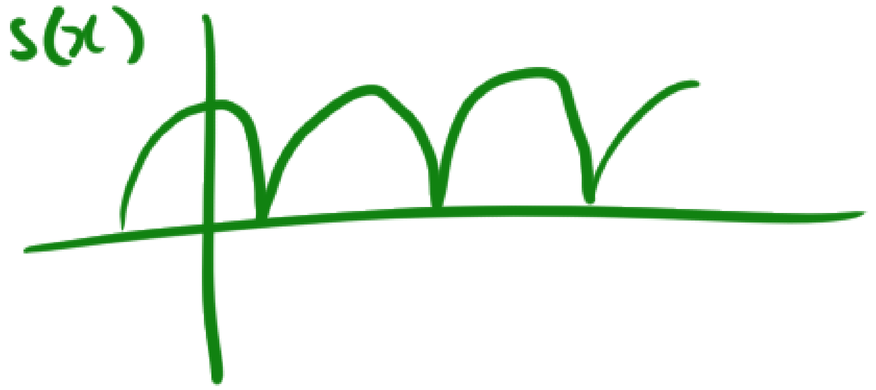
$f(x) = x^2$

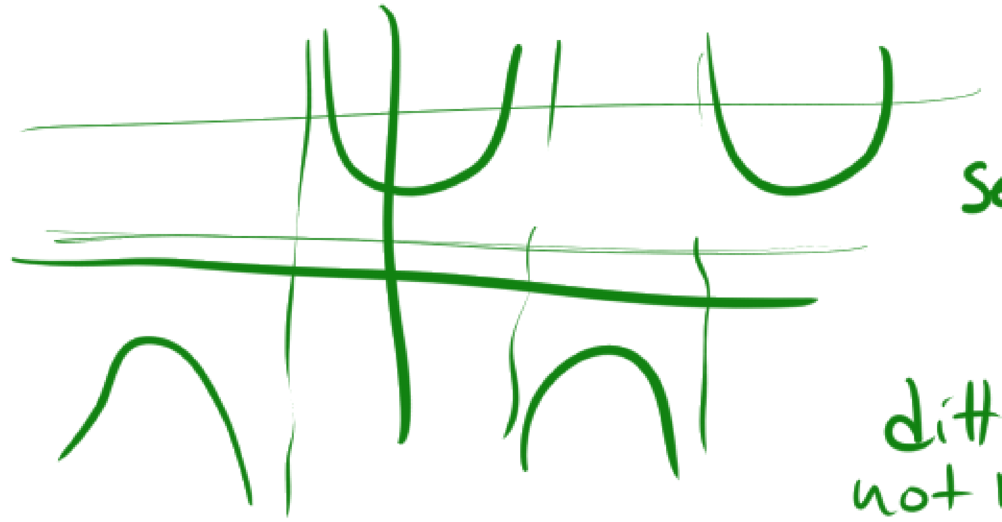


$f(x) = \cos(x)$



$\cos^2(x)$





$\sec(x)$

ditto
not $\mathbb{R} \to \mathbb{R}$

Let $f : \mathbb{R} \longrightarrow \underbrace{\mathbb{R}^{>0}}_{\{x \in \mathbb{R} : x > 0\}}$ given by $f(x) = e^x$. Then f is bijective.

<u>Proof.</u> First, we will prove surjectivity.

Let $y \in \mathbb{R}^{>0}$.

(natural log)

Then let $x = \ln(y)$.

Then $\ln(y) \in \mathbb{R}$ and $f(\ln(y)) = e^{\ln(y)} = y$.

Second, we will prove injectivity.

Let $x_1, x_2 \in \mathbb{R}$.

Assume $f(x_1) = f(x_2)$.

Then $e^{x_1} = e^{x_2}$. Note $e^{x_1}, e^{x_2} > 0$.

$\Rightarrow \ln(e^{x_1}) = \ln(e^{x_2})$

$\Rightarrow x_1 = x_2$.

<u>Note:</u> We used $\ln$

This is an "inverse" for e^x.

Definitions. The function $f: A \to A$ given by $f(a) = a$ is called the identity function.