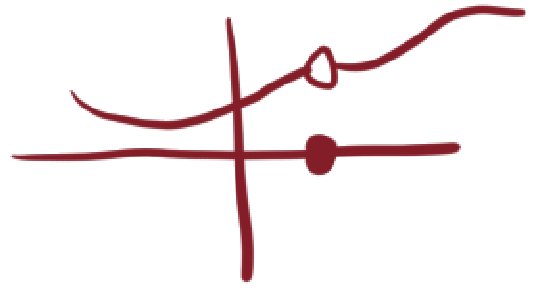
Definition. Let $f: \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$. We say f is continuous at $x = a$ if for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that

$$|f(a) - f(b)| < \varepsilon \text{ whenever } |a - b| < \delta.$$

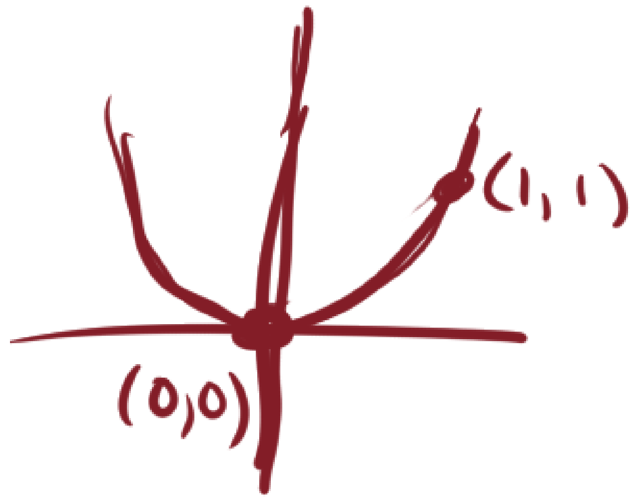
$$\boxed{\forall \varepsilon > 0, \exists \delta > 0, \; |a-b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon.}$$

challenge — response — if I'm close to a — then — the result is close to f(a)

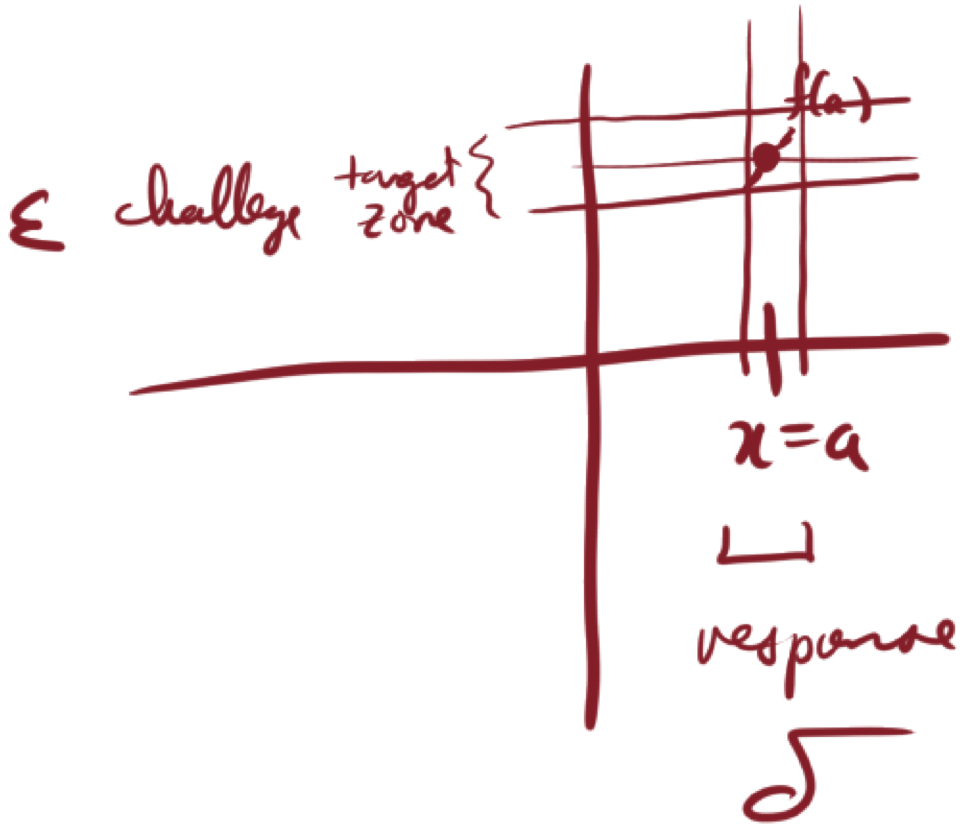


Example. $f(x) = x^2$; is this continuous at $x = 0$?

$f(0) = 0^2 = 0$



challenge	response	verification
$\varepsilon = 1$	$\delta = 1$	$\lvert b\rvert < 1 \Rightarrow \lvert b^2\rvert < 1$
$\varepsilon = 1/4$	$\delta = \frac{1}{2}$	$\lvert b\rvert < \frac{1}{2} \Rightarrow \lvert b^2\rvert < 1/4$
$\varepsilon = 1/9$	$\delta = \frac{1}{3}$	
ε	$\sqrt{\varepsilon}$	$\boxed{\lvert b\rvert < \sqrt{\varepsilon} \Rightarrow \lvert b^2\rvert < \varepsilon}$



a is 0.

Theorem. $f(x) = x^2$ is continuous at $x = 0$.

Proof. Let $\varepsilon > 0$.

Let $\delta = \sqrt{\varepsilon}$.

Then suppose $|0 - b| < \delta$

Then $|b| < \delta = \sqrt{\varepsilon}$.

So $|b^2| < \varepsilon$.

So $|f(0) - f(b)| < \varepsilon$. $\blacksquare$

challenge

response

verification: prove that

$\boxed{\text{"if } |a-b| < \delta \text{ then } |f(a) - f(b)| < \varepsilon\text{"}}$

Functions!

Def^n. A function $f: A \to B$ is way to assign one value of B to each value of A.

A: "domain"
B: "codomain"

We write $f(a) = b$ when f assigns the value b to the input a.

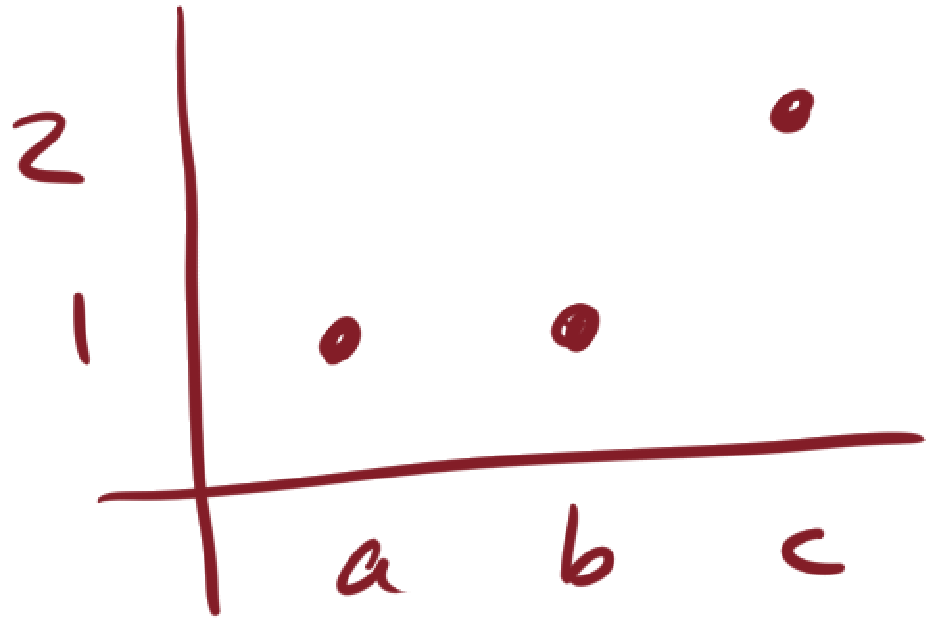
Example.

$f: \{a, b, c\} \to \{1, 2\}$ given by $\underbrace{\boxed{f(a) = f(b) = 1 \text{ and } f(c) = 2.}}_{}$ data of f

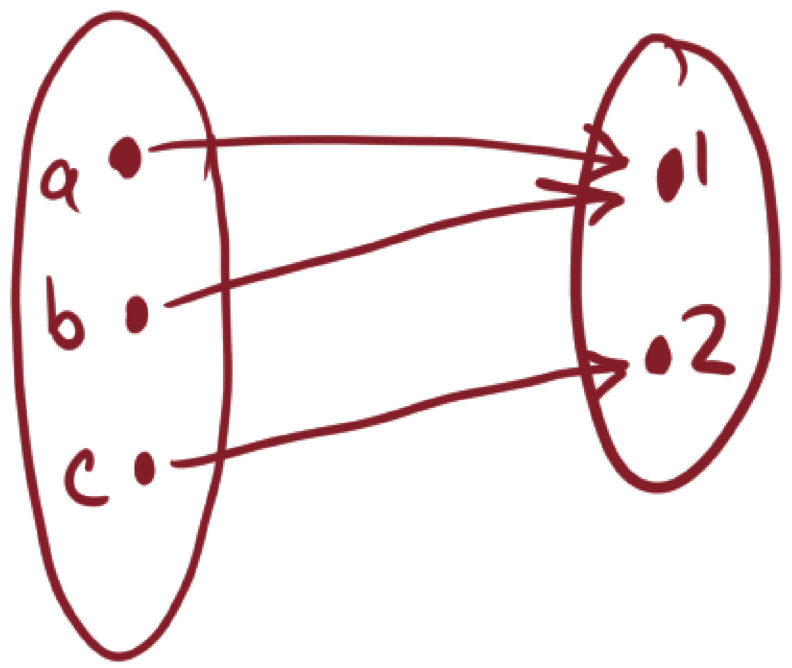
Table:

x	$f(x)$
a	1
b	1
c	2

"Graph"



Arrow Diagram



failure of injectivity

Ordered Pairs.

$$\{(a, 1), (b, 1), (c, 2)\}$$

"list of data"

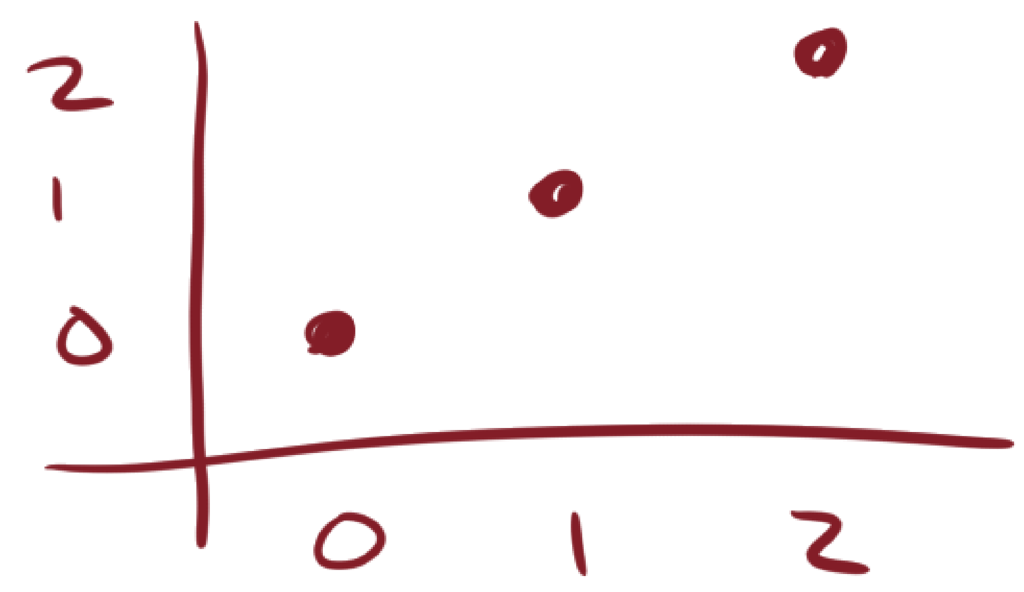
Formal Def^n A function $f: A \to B$ is a subset of $A \times B$
"from A to B"
such that for each $a \in A$, there is exactly one pair in f with 1st element equal to a.

Example. $f: \{0,1,2\} \to \{0,1,2\}$ given by $f(x) = x$.

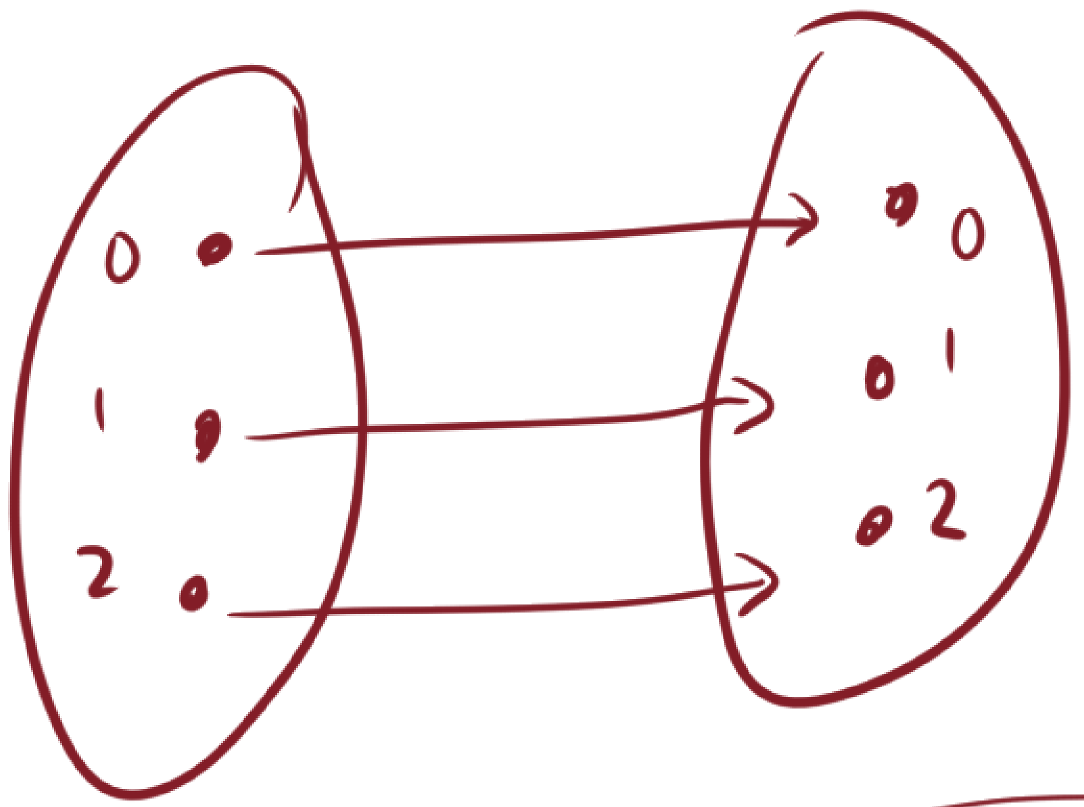
Table.

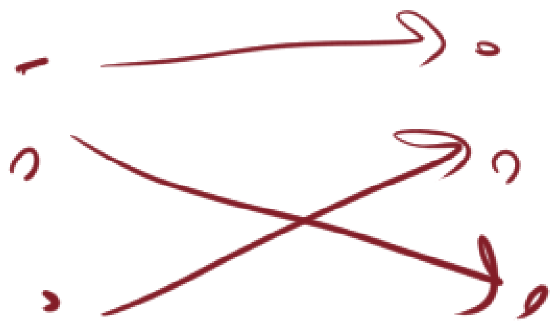
x	$f(x)$
0	0
1	1
2	2

Graph



Arrow Diagram





Set of Pairs

$$\{(0,0), (1,1), (2,2)\}$$

Def^n. A function $f: A \to B$

1. is <u>surjective</u> if for every $b \in B$, there exists an $a \in A$ such that $f(a) = b$. ("onto")

2. is <u>injective</u> if for every pair $a_1, a_2 \in A$, if $a_1 \neq a_2$ then $f(a_1) \neq f(a_2)$ "no collisions" ("1-to-1")

   [Equivalently, if for every pair $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$]

3. is <u>bijective</u> if it is both injective and surjective.

Ex. $f: \{a, b, c\} \to \boxed{\{1, 2\}}$

x	$f(x)$
a	1
b	1
c	2

challenge	response	verification
1	a or b	$f(a) = 1$, $f(b) = 1$
2	c	$f(c) = 2$

f <u>is</u> surjective because every output <u>occurs</u> for some input

f is <u>not</u> injective because $f(a) = f(b)$ but $a \neq b$.

Ex. $g: \{a, b\} \to \{1, 2, 3\}$

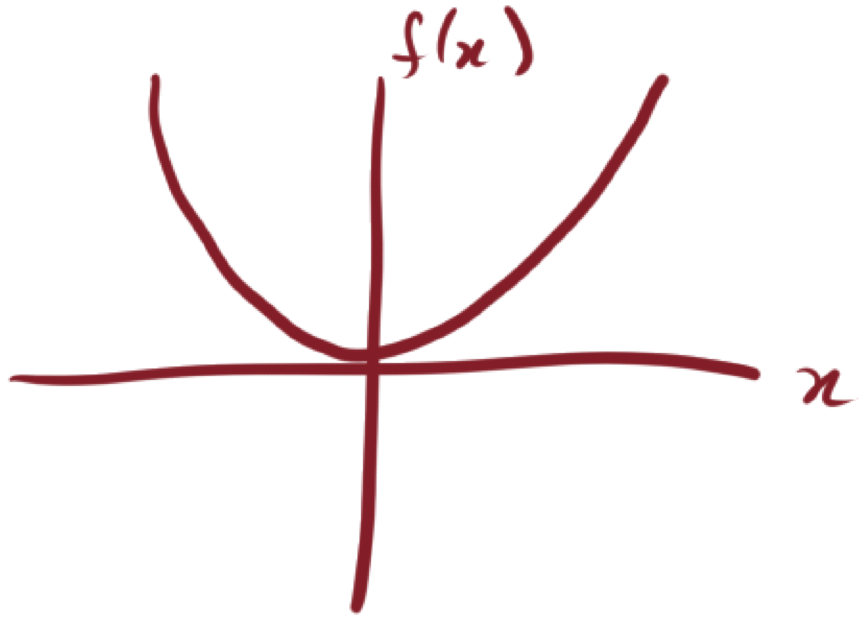
x	$f(x)$
a	1
b	2

g is <u>not</u> surjective because 3 does not occur.

g is injective, because there are no collisions.

Example

$f: \overset{\text{domain}}{\mathbb{R}} \to \overset{\text{codomain}}{\mathbb{R}}$, $\quad f(x) = x^2$.



injective?

$\boxed{\text{NO}}$

$$\left.\begin{aligned} f(-1) &= (-1)^2 = 1 \\ f(1) &= 1^2 = 1 \end{aligned}\right\} \text{ collision}$$

surjective?

$\boxed{\text{NO}}$

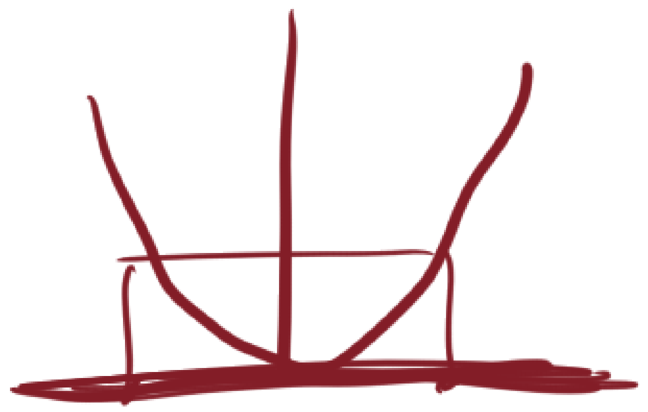
There does not exist any $x \in \mathbb{R}$ s.t. $x^2 = -1$.

So -1 does not occur as an output.

bijective? NO

Modification:

$f: \mathbb{R} \to \{x \in \mathbb{R} : x \geq 0\}$, $\quad f(x) = x^2$



injective? $\boxed{\text{NO}}$

surjective? $\boxed{\text{YES}}$ (every non-negative real # is a square of a real number)

bijective? $\boxed{\text{NO}}$