Definition. Let $L_1, L_2, \ldots, L_n, \ldots$ be a sequence of real numbers. We say the sequence L_n converges to a real number L if for every $\varepsilon > 0$, there exists a positive integer N so that $|L_n - L| < \varepsilon$ for all $n > N$.

"You can get $\underbrace{\text{as close as you want}}_{\text{challenge}}$ (ε) if you $\underbrace{\text{wait long enough (N)}}_{\text{response}}$."

$$\boxed{\forall\, \varepsilon > 0,\ \exists\, N \in \mathbb{N},\ n > N \Rightarrow |L_n - L| < \varepsilon}$$

Example. $L_1 = 1,\ L_2 = 1/2,\ L_3 = 1/3,\ L_4 = 1/4, \ldots, L_n = 1/n, \ldots$, $L = 0$.

challenge	response	verify
$\varepsilon = 1$	$N = 1$	$n > 1 \Rightarrow \lvert 1/n \rvert < 1$
$\varepsilon = 1/2$	$N = 2$	$n > 2 \Rightarrow \lvert 1/n \rvert < \frac{1}{2}$.
$\vdots$	$\vdots$	$\vdots$
$\varepsilon = 1/m$	$N = m$	$n > m \Rightarrow \lvert 1/n \rvert < \frac{1}{m}$.
ε	any $N > \frac{1}{\varepsilon}$	$n > N \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \lvert \frac{1}{n} \rvert < \varepsilon$

Definition. Let $L_1, L_2, \ldots, L_n, \ldots$ be a sequence of real numbers. We say the sequence L_n <u>converges</u> to a real number L if for every $\varepsilon > 0$, there exists a positive integer N so that $|L_n - L| < \varepsilon$ for all $n > N$.

Thm. $L_n = \frac{1}{n}$ converges to $L = 0$.

Pf. Let $\varepsilon > 0$.

challenge

Let N be any integer greater than $\frac{1}{\varepsilon}$.

response

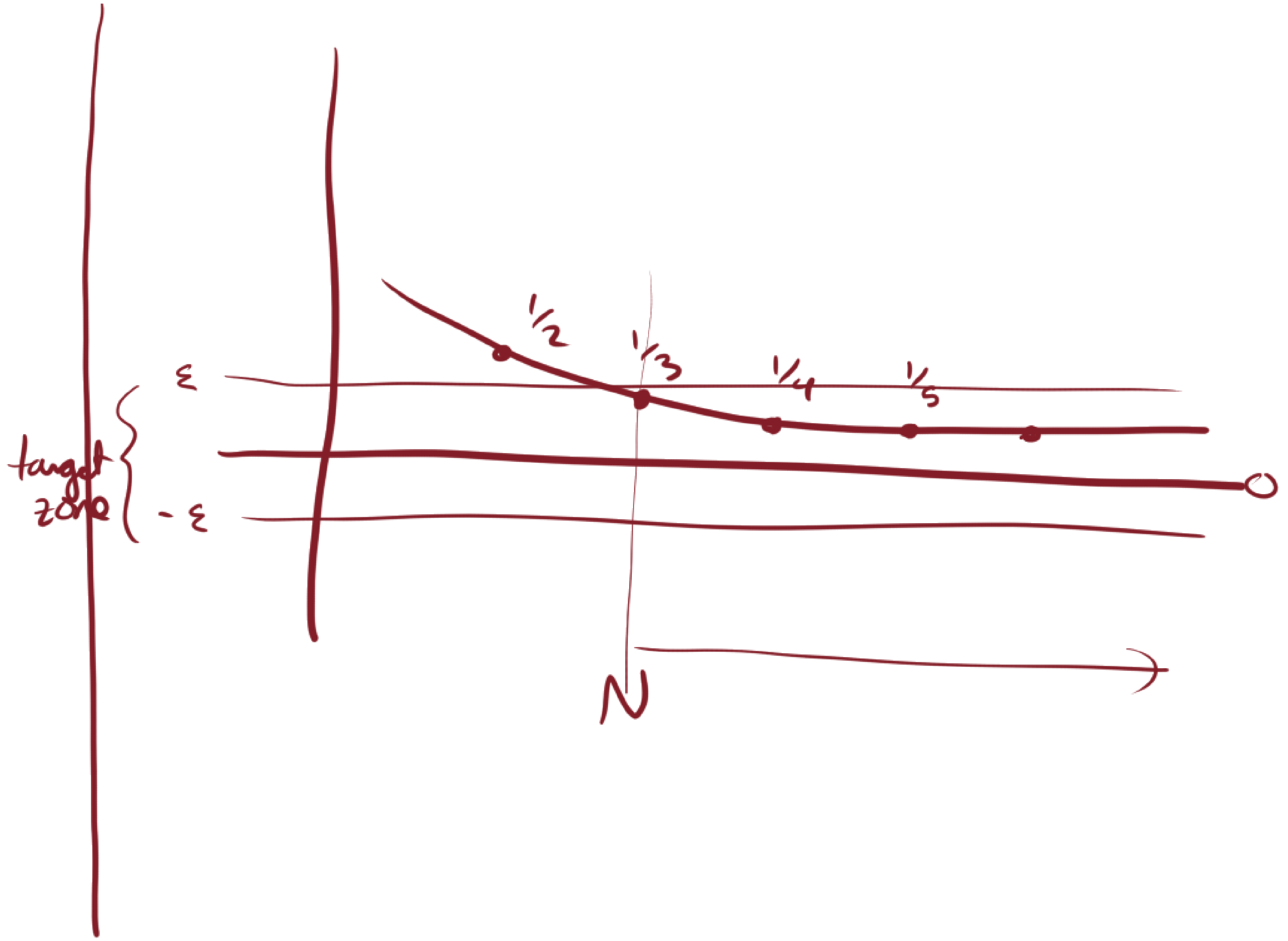
verification

Suppose $n > N$.

Then $n > \frac{1}{\varepsilon}$.

Then $\left|\frac{1}{n}\right| < \varepsilon$.

Therefore $|L_n - L| < \varepsilon$. $\boxtimes$



Scratch: ε | any $N > \frac{1}{\varepsilon}$ | $n > N \Rightarrow n > \frac{1}{\varepsilon} \Rightarrow \left|\frac{1}{n}\right| < \varepsilon$

Definition. Let $f: \mathbb{R} \to \mathbb{R}$ be a function, and let $a \in \mathbb{R}$. We say f is continuous at $x = a$ if for every real $\varepsilon > 0$, there exists a real $\delta > 0$ such that

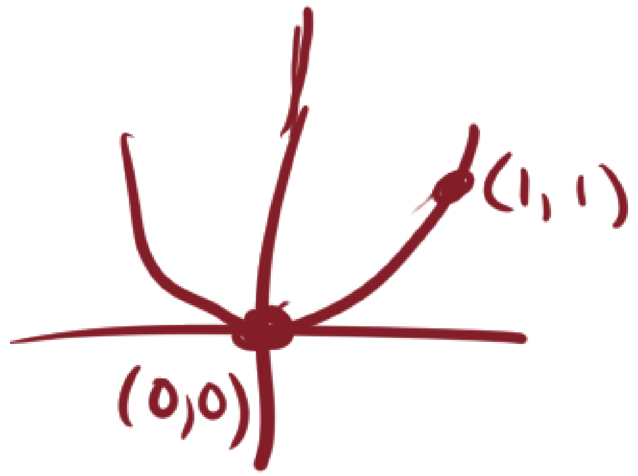
$$|f(a) - f(b)| < \varepsilon \text{ whenever } |a - b| < \delta.$$

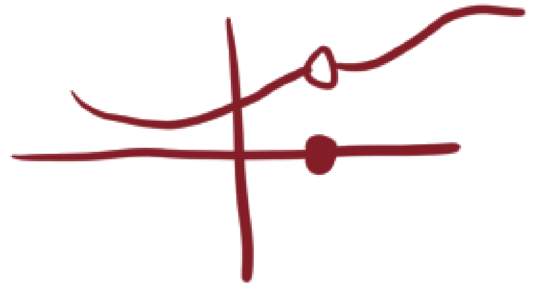
$$\boxed{\forall\, \varepsilon > 0,\ \exists\, \delta > 0,\ \underline{|a-b| < \delta \Rightarrow |f(a) - f(b)| < \varepsilon}.}$$

challenge   response   if I'm close to a   then   the result is close to f(a)

Example. $f(x) = x^2$: is this continuous at $x = 0$?

$f(0) = 0^2 = 0$





challenge	response	verification
$\varepsilon = 1$	$\delta = 1$	$\lvert b\rvert < 1 \Rightarrow \lvert b^2\rvert < 1$
$\varepsilon = 1/4$	$\delta = \frac{1}{2}$	$\lvert b\rvert < \frac{1}{2} \Rightarrow \lvert b^2\rvert < 1/4$
$\varepsilon = 1/9$	$\delta = \frac{1}{3}$	
ε	$\sqrt{\varepsilon}$	$\lvert b\rvert < \sqrt{\varepsilon} \Rightarrow \lvert b^2\rvert < \varepsilon$