Theorem. There exist irrational numbers a and b such that a^b is rational.

Pf. Suppose $x = \sqrt{2}$ and $y = \sqrt{2}$. These are irrational.

Case 1: If x^y is rational, then we have found an example:

$$a = \sqrt{2}, \quad b = \sqrt{2}.$$

Case 2: If x^y is irrational, then we have found an example:

$$a = x^y = \sqrt{2}^{\sqrt{2}}, \quad b = \sqrt{2}$$

and $a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2.$

$\square$

If $a = \sqrt{2}$, $b = \sqrt{2}$, a^b rational

done ✓

What if a^b is irrational?

Let $a = \sqrt{2}^{\sqrt{2}}$ irr
$b = \sqrt{2}$

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^2 = 2.$$

rational ✓

Theorem. For all nonzero $x \in \mathbb{R}$, there exists a $y \in \mathbb{R}$ such that $xy = 1$.

Examples. 1) If $x = 1$? $y = 1$.

2) If $x = -8$? $y = \frac{1}{-8} = \frac{-1}{8}$.

Pf. Let $x \in \mathbb{R}$ be non-zero.

Let $y = 1/x$.

Then $y \in \mathbb{R}$ and $xy = 1$. $\boxtimes$

Theorem. Between any two rational numbers, there is an irrational number.

Experiment.

Challenge	Response
between 0 and 100	$\sqrt{2}$
between 100 and 200	$100+\sqrt{2}$
between 0 and 1	$\sqrt{2}-1$, $\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$
between 0 and $\frac{1}{100}$?	$\frac{1}{100}\cdot\frac{1}{\sqrt{2}}$

Useful Facts:

1. $\sqrt{2}$ is irr
2. rat + irr = irr
3. rat $\bullet$ irr = irr

Thm. For $x, y \in \mathbb{Q}$ such that $x<y$, there exists an irrational number r such that $x<r<y$.

$$\forall x, y \in \mathbb{Q},\ \left(x<y \Longrightarrow \left(\exists\, r \in \mathbb{R}-\mathbb{Q},\ x<r<y\right)\right).$$

Theorem. Between any two rational numbers, there is an irrational number.

Pf. Let $x, y \in \mathbb{Q}$ s.t. $x < y$. Let $\ell = y - x$ (the length of the interval), which is rational.

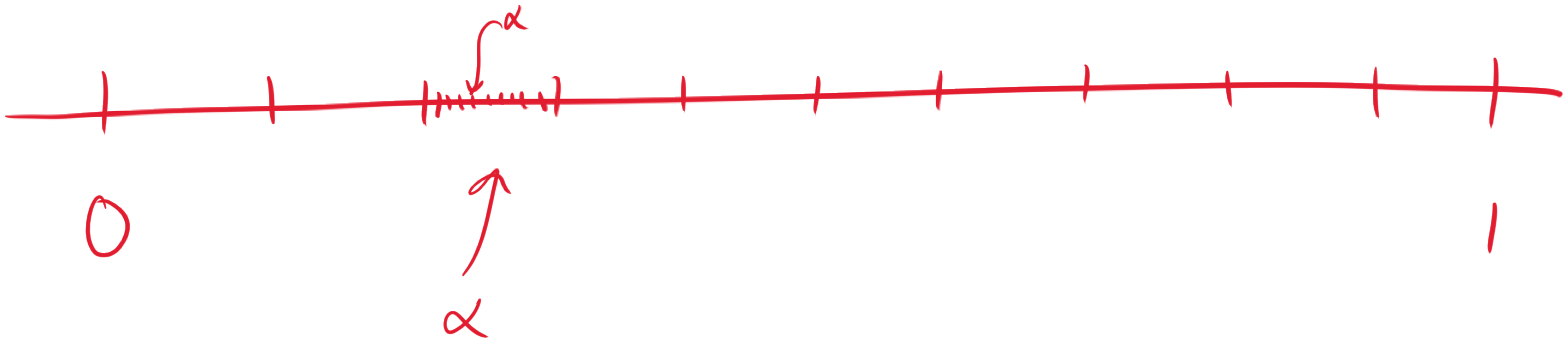
Note that $\frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$ is irrational, as the product of an irrational and a rational.

And $0 < \frac{1}{\sqrt{2}} < 1$.

Therefore $0 < \frac{\ell}{\sqrt{2}} < \ell$. And $\frac{\ell}{\sqrt{2}}$ is irrational.

Finally, $x < \frac{\ell}{\sqrt{2}} + x < \ell + x = y$.

Since x is rational, $\frac{\ell}{\sqrt{2}} + x$ is irrational. This is the number we seek. $\blacksquare$



$$\alpha = 0.2345179\ldots.$$