Boolean Algebra

"if and only if"

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

Boolean Algebra Identities

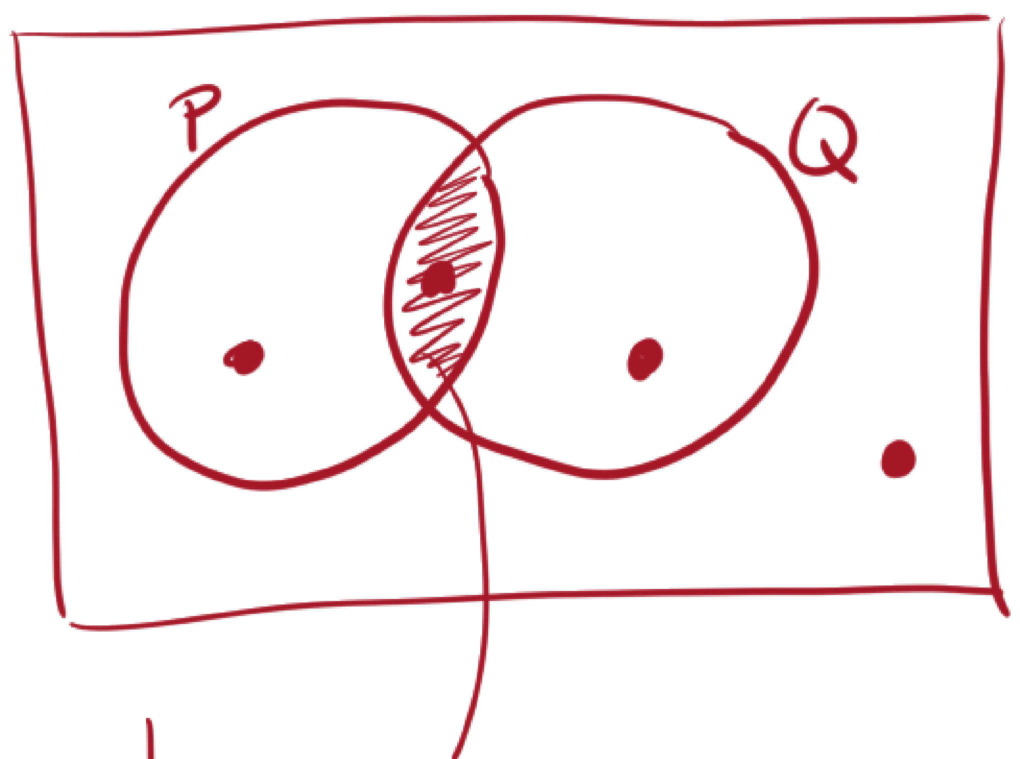
"identity"

$3x + x = 4x$

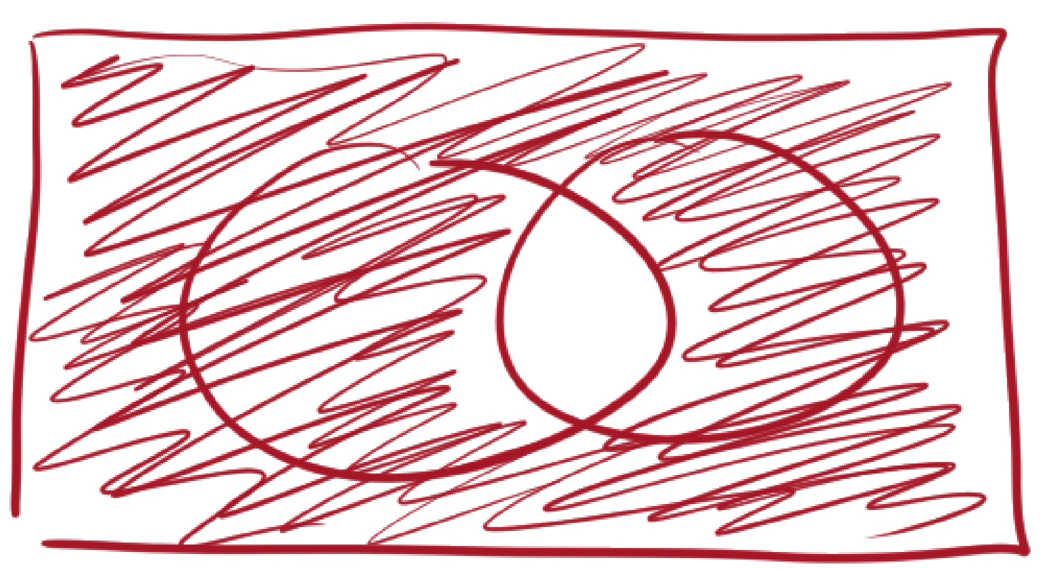
De Morgan's Laws

$\sim(P \wedge Q) = (\sim P) \vee (\sim Q)$

$\sim(P \vee Q) = (\sim P) \wedge (\sim Q)$



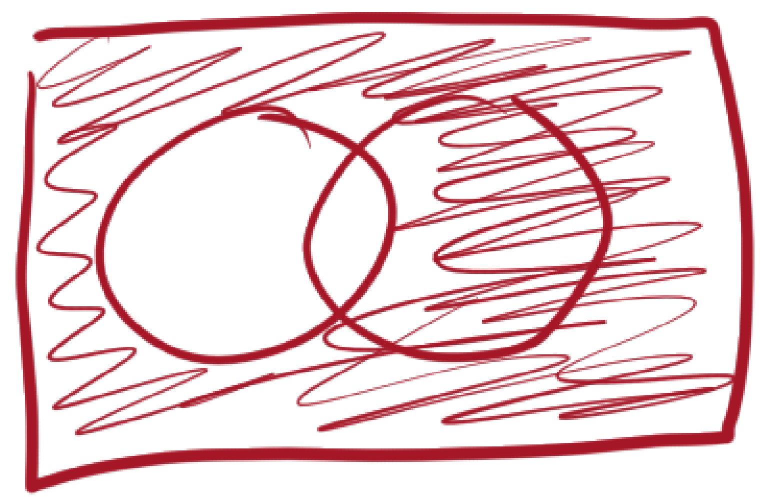
$P \wedge Q$



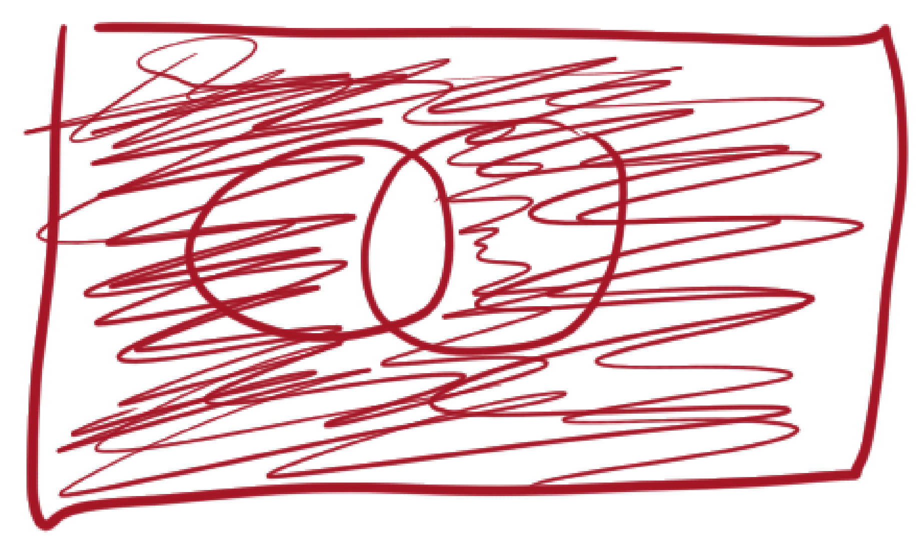
$\sim(P \wedge Q)$

=

$\sim P$



$\sim Q$



$(\sim P) \vee (\sim Q)$

Exercise

Do this one via Venn Diagrams.

Commutative Laws:

$$P \wedge Q = Q \wedge P$$
$$P \vee Q = Q \vee P$$

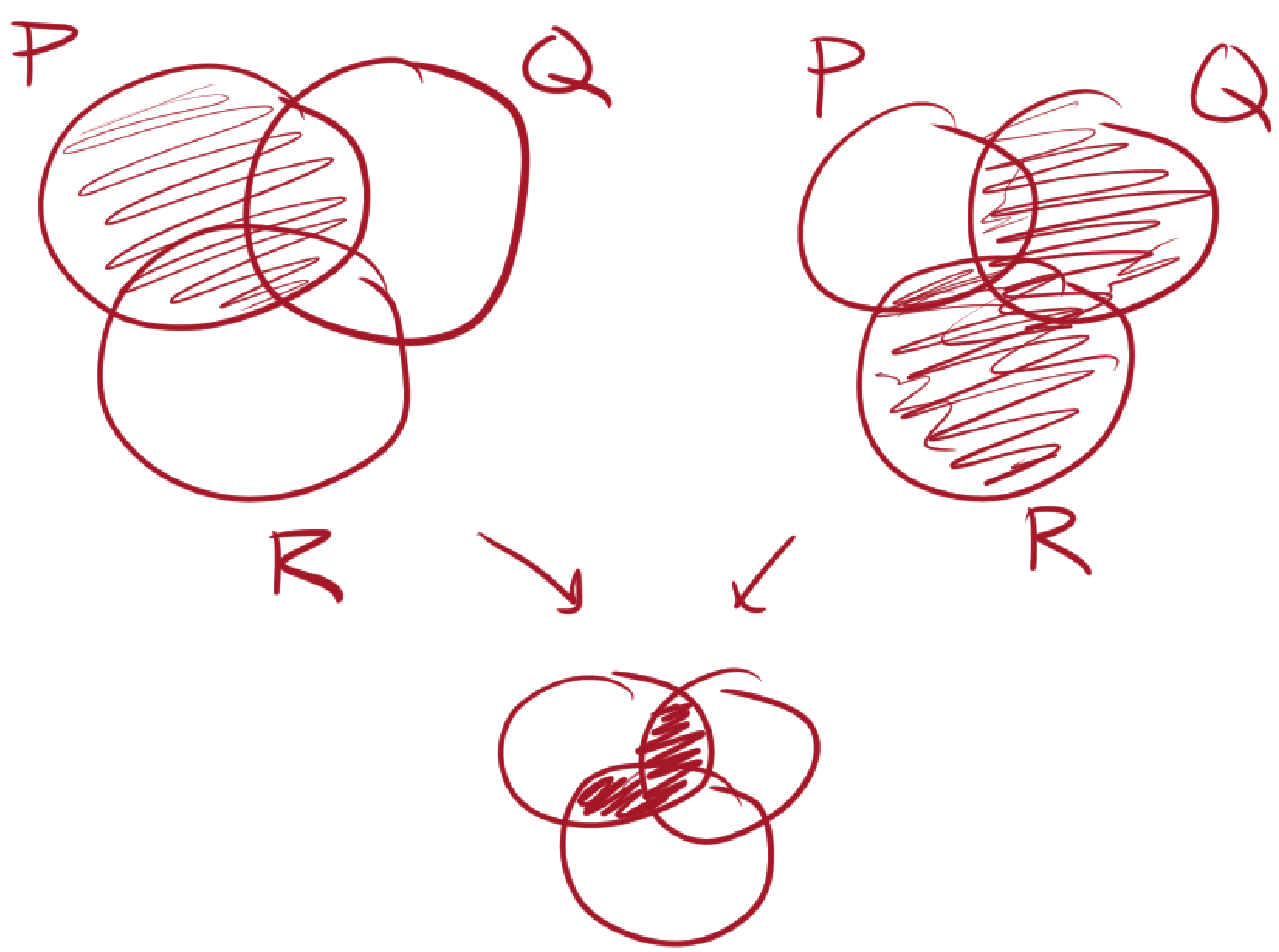
Note: $P \Rightarrow Q \neq Q \Rightarrow P$

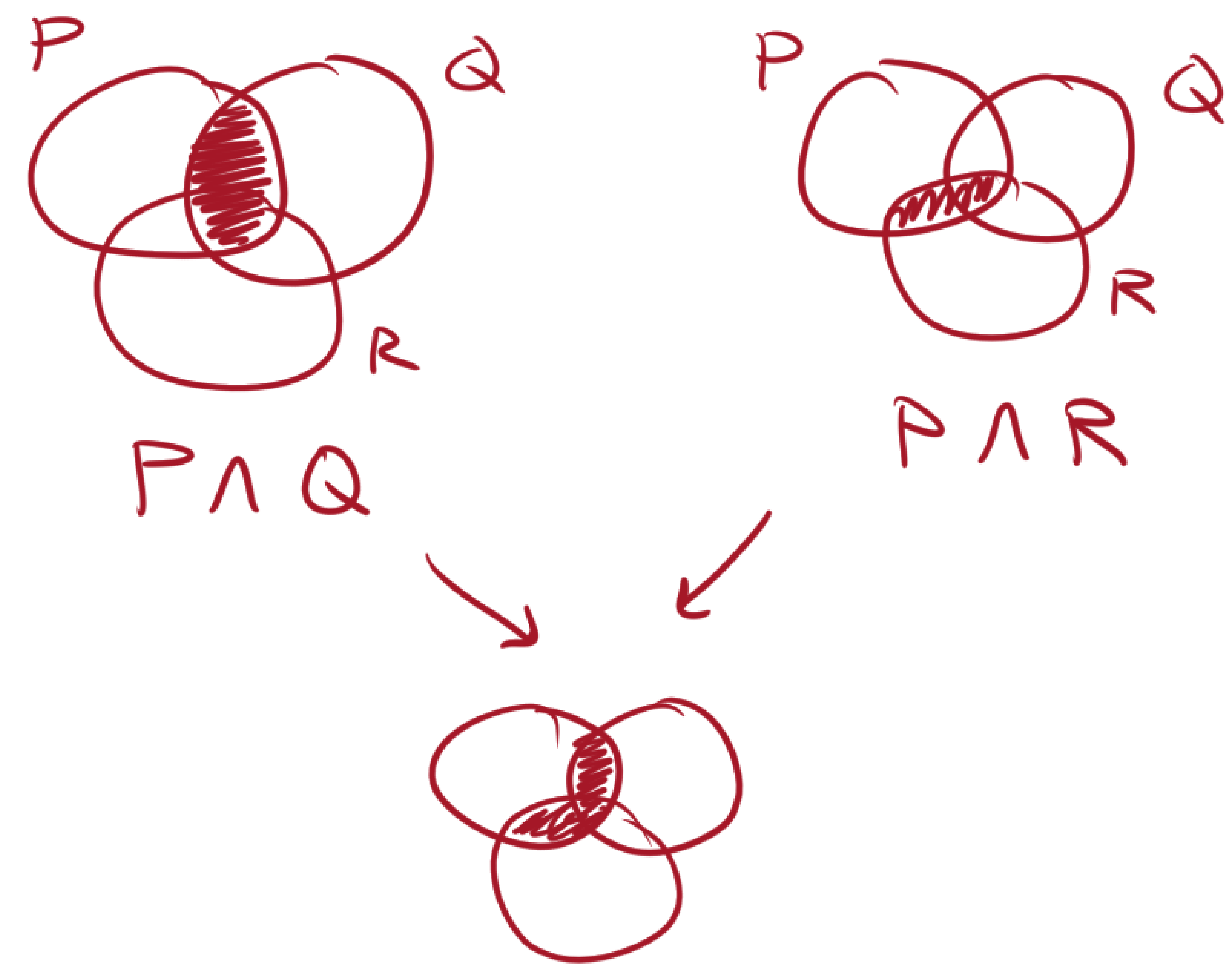
Distributive Laws:

$$\rightarrow P \wedge (Q \vee R) = (P \wedge Q) \vee (P \wedge R)$$

$$P \vee (Q \wedge R) = (P \vee Q) \wedge (P \vee R)$$

Idea: $a(b+c) = ab + ac$





numbers:

$(ab)c = a(bc)$

Associative Laws: $P \wedge (Q \wedge R) = (P \wedge Q) \wedge R$

$P \vee (Q \vee R) = (P \vee Q) \vee R$

Contrapositive Law: $P \Rightarrow Q = (\sim Q) \Rightarrow (\sim P)$

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

Negation of Implies: $\sim (P \Rightarrow Q) = P \wedge (\sim Q)$

Examples: If x is divisible by 4 then x is even.

Contrapositive: If x is odd then x is not divisible by 4.

another equivalent formulation: There is no integer x which is divisible by 4 but is odd.

no x s.t. P and $\sim Q$

Contrapositive Proof

Direct Proof

Thm. If A then B.

Pf. Assume A.

$\vdots$

Prove B.

Contrapositive Proof

If $\sim B$ then $\sim A$.

Thm. If A then B.

Pf. Assume $\sim B$.

$\vdots$

Prove $\sim A$.

Scratch:

$n = 2k+1$

$\big\downarrow$ square

$n^2 = 2(\quad)+1$

Example. $\boxed{\text{If } n^2 \text{ is even, then } n \text{ is even.}}$

Direct: Assume n^2 is even.

$\vdots$

Therefore n is even.

scratch

$n^2 = 2k$

$\vdots$

$n = 2\,?$

Contra: Assume n is odd.

Then $n = 2k+1$.

So $n^2 = (2k+1)^2$

$= 2(2k^2+2k)+1$

Therefore n^2 is odd.

<u>Def</u>

<u>Def^n</u>. A number $x \in \mathbb{R}$ is called <u>rational</u> if $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$.
Otherwise x is called <u>irrational</u>.

$\frac{1}{3}$ is rational since $\frac{1}{3} = \frac{a}{b}$ w/ $a = 1$, $b = 3$.

<u>Thm</u>. If x and y are rational, so is xy.

<u>Pf.</u> Suppose $x = a/b$, $y = c/d$, for some $a, b, c, d \in \mathbb{Z}$.
Then $xy = \frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$. Since $ac, bd \in \mathbb{Z}$,
xy is rational. $\boxtimes$

<u>Thm</u>. If x and y are rational, so is $x+y$.

$\rightarrow$ Rationals, $\mathbb{Q}$, are closed under addition & multiplication.

Def^n. A number $x \in \mathbb{R}$ is called <u>rational</u> if $x = \frac{a}{b}$ for some $a, b \in \mathbb{Z}$.
Otherwise x is called <u>irrational</u>.

Thm. If y^3 is irrational, then y is irrational.

Pf. We will prove this by contrapositive.
Assume y rational.
So $y = a/b$ for some $a, b \in \mathbb{Z}$.
So $y^3 = (a/b)^3 = \frac{a^3}{b^3}$.
So y is rational. $\boxtimes$

$$\frac{a}{b} = \frac{a'}{b'}$$

$$\frac{1}{3} = \frac{2}{6}$$

Proof by Contradiction

Thm. There are no integer solutions to $2x + 4y = 1$.

Pf. Suppose for a contradiction, that $x, y \in \mathbb{Z}$ are a solution.

Then $2(x + 2y) = 1$, and $x + 2y \in \mathbb{Z}$.

Then 1 is even.

This is a contradiction. $\blacksquare$