Counting: Correcting Overcounting

Q1: How many ways to make a 4-letter word w/ scrabble tiles A B C D?

Problem B

$$4 \cdot 3 \cdot 2 \cdot 1$$

↑ 1st letter ↑ 2nd ↑ 3rd ↑ 4th letter

Q2: How many ways - - - - AACD?

Problem A

$\frac{4 \cdot 3 \cdot 2 \cdot 1}{2}$ words w/ A_1, A_2, C, D

overcounting { 2 : but A_2A_1CD and A_1A_2CD → same word

Principle of Overcounting (Division Principle)

If counting Problem A relates to counting Problem B in such a way that each item in Problem A corresponds to exactly n items in Problem B

Then

$$(\#\text{ solns in Prob A}) = \frac{(\#\text{ Soln in Prob B})}{n}.$$

Fundamental Examples

Q: How many ways to order n items?

(n stages)

$$n! = n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot 1$$

Q: How many ways to choose & order k out of n items?

(k stages)

$$n \cdot (n-1) \cdot (n-2) \cdot \cdots \cdot (n-k+1) = \frac{n!}{(n-k)!}$$

eg. 3 of 8 items

$$8 \cdot 7 \cdot 6 = \frac{8!}{5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

overcounting

Q: How many ways to choose k out of n items?

$$= \frac{(\#\text{ ways to choose \& order})}{(\#\text{ ways to order})} = \frac{n!/(n-k)!}{k!} = \frac{n!}{(n-k)!\,k!} =: \binom{n}{k}$$

Notation

"n choose k"

Binomial Coefficient.

Addition Principle

If the ways a task can be accomplished falls into disjoint cases, then

$$\#\text{ways} = \left(\begin{array}{c}\#\text{ways in} \\ \text{case 1}\end{array}\right) + \left(\begin{array}{c}\#\text{ways} \\ \text{in case 2}\end{array}\right) + \cdots \text{ etc.}$$

Definition. Two sets are disjoint if they share no elements.

i.e. A and B are disjoint if $A \cap B = \emptyset$.

Exampl. Q: How many ways can you either roll a die or toss a coin?

$$\underset{\text{die}}{6} + \underset{\text{coin}}{2} = 8$$

Q: How many ways can you form a committee of 2, 3 or 4 people from a room of 30 ppl?

$$\binom{30}{2} + \binom{30}{3} + \binom{30}{4}$$

Theorem. $\sum_{k=0}^{n} \binom{n}{k} = 2^n$.

$\binom{n}{k} = \frac{n!}{k!(n-k)!}$

Pf. (Combinatorial Proof)

Problem → Let us consider the problem of counting the # of subsets of a set A of n elements.

Answer #1 → We know by a previous theorem that this is $|\mathcal{P}(A)| = 2^{|A|} = 2^n$.

Answer #2 → We will give another answer:

Let us count the subsets in cases according to cardinality.

The # of subsets of size k is $\binom{n}{k}$.

The possible cardinalities are $0, 1, \ldots, n$.

By the addition principle, the # of subsets of A is $\sum_{k=0}^{n} \binom{n}{k}$.

Therefore these are the same. → Since these are 2 correct answers to the same problem, they must be equal. ∎

Binomial Theorem $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$ (✱)

eg. $(x+y)^2 = x^2 + 2xy + y^2$

$1 = \binom{2}{0}$ $2 = \binom{2}{1}$ $1 = \binom{2}{2}$

Proof. When we are expanding the multiplication

$$\underbrace{(x+y)(x+y)\cdots\cdots(x+y)}_{n \text{ times}}$$

we obtain one term for every way we can choose x or y in each parenthesis (for example, choosing all x's gives x^n).

The # of times we obtain $x^k y^{n-k}$ is $\binom{n}{k}$ since we must choose x k times and y $n-k$ times, i.e. pick k of the parentheses to designate "x".

Therefore the result is $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$. ☐

Corollary.

$$\sum_{k=0}^{n} \binom{n}{k} = 2^n$$

Pf. Let $x = y = 1$ in Binomial Theorem. ☐

Corollary. $n 2^{n-1} = \sum_{k=0}^{n} k \binom{n}{k}$.

Pf. Take derivative of (✱) w.r.t. x

$$n(x+y)^{n-1} = \sum_{k=0}^{n} k \binom{n}{k} x^{k-1} y^{n-k}.$$

Now set $x = y = 1$. ☐