Def^n. Let $a, b \in \mathbb{Z}$. We say a divides b, written $a \mid b$, if there exists some $k \in \mathbb{Z}$ such that $b = ka$.

$a \mid b$ — verb — sentence

$\frac{a}{b}$ — number

$0 = 0 \cdot 3$ ✓ $\quad 3 \mid 0$. ✓

Theorem. If $\underbrace{a \mid b \text{ and } a \mid c}_{\text{HYPOTHESIS}}$ then $\underbrace{a \mid b+c}_{\text{CONCLUSION}}$.

Pf. Assume $a \mid b$ and $a \mid c$.

Then $b = ka$ for some $k \in \mathbb{Z}$.
and $c = la$ for some $l \in \mathbb{Z}$.

Therefore

$$\begin{aligned} b+c &= ka + la \\ &= (k+l)a. \end{aligned}$$

Therefore $b+c = ma$ for some $m \in \mathbb{Z}$.
So $a \mid b+c$. $\blacksquare$

scratchwork

$a \mid b$ becomes $b = ka$
$a \mid c$ ... $c = la$
(know)

$a \mid b+c$ ... $b+c = ma$ for some m (GOAL)

$$\begin{aligned} b+c &= ka + la \\ &= \underbrace{(k+l)}_{\text{my } m} a \end{aligned}$$

"direct proof"

Theorem. $\{x \in \mathbb{Z} : 6 \mid x\} \subseteq \{x \in \mathbb{Z} : 2 \mid x\}$

Pf. The left-hand set is the set of integers divisible by 6, in other words, multiples of 6.

The right-had set, similarly, is the set of multiples of 2.

To check set containment, let us consider an element of the left-hand set.

It can be expressed as $6k$ for some $k \in \mathbb{Z}$.

But $6k = 2(3k)$.

So this element is also a multiple of 2, hence contained in the right-hand set. $\boxtimes$

Scratch.

$A \subseteq B$

means

any elt of A is an elt of B

$\Updownarrow$

any multiple of 6 is a multiple of 2

$$\boxed{6k = 2 \cdot (3k)}$$

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But $6k = 2(3k)$.

So this element is also a multiple of 2, hence contained in the right-hand set. $\blacksquare$

$$6 = 2+2+2 = 3 \cdot 2 = \sum_{i=0}^{2} 2.$$

Scratch.

$A \subseteq B$ means any elt of A is an elt of B

$\updownarrow$

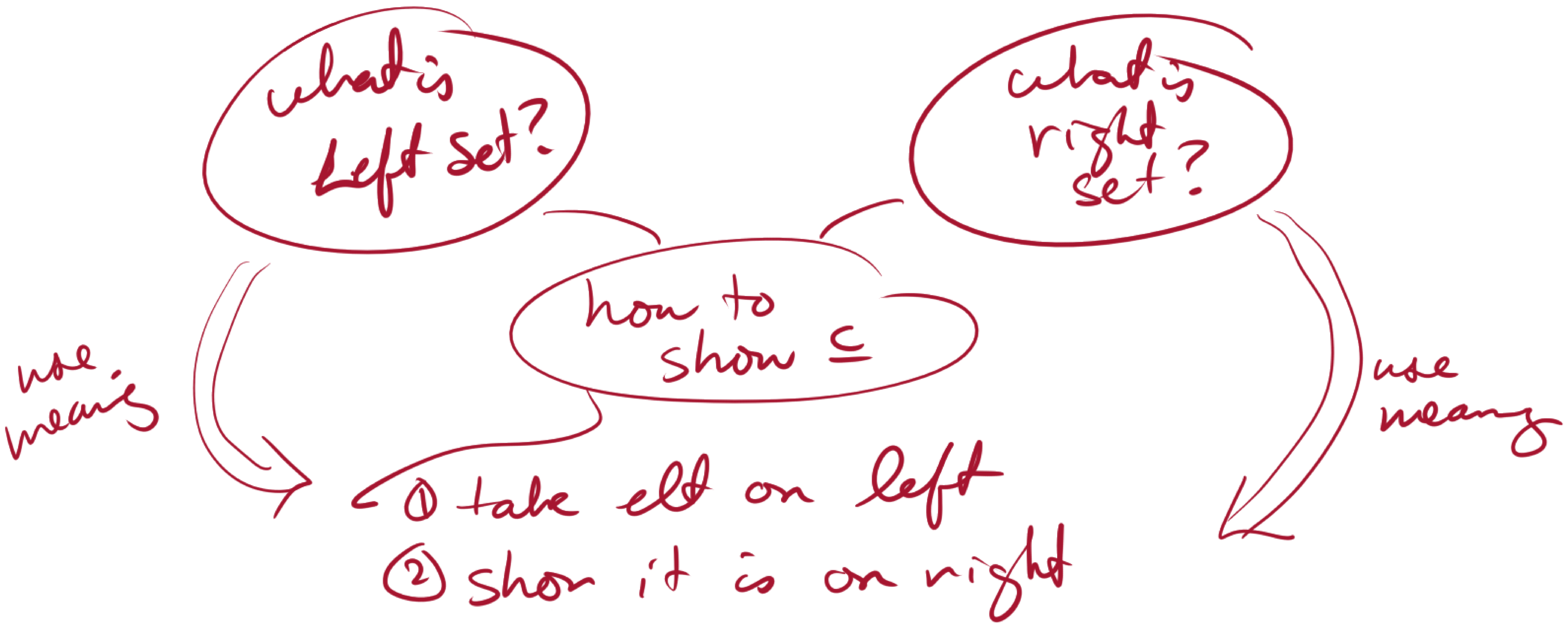
any multiple of 6 is a multiple of 2

$$\boxed{6k = 2\cdot(3k)}$$

<u>Pf.</u> We must show that any multiple of 6 is a multiple of 2.

But this is clear since $6k = 2(3k)$.

∎



Pf. The theorem can be restated as

"$\{x \in \mathbb{Z} : x = 6k \text{ for some } k \in \mathbb{Z}\} \subseteq \{x \in \mathbb{Z} : x = 2\ell \text{ for some } \ell \in \mathbb{Z}\}$"

It could be further restated as

"any $x \in \mathbb{Z}$ of the form $6k$ for some $k \in \mathbb{Z}$ is of the form 2ℓ for some $\ell \in \mathbb{Z}$".

But this is clear since $6k = 2(3k)$, so we can take $\ell = 3k$.

$\blacksquare$