Practice in $\mathbb{F}_3[x]$.

1. $(x^2+2x+1)(x+1) = x^3+2x^2+x+x^2+2x+1 = x^3+1$

2.

$$
\begin{array}{r|l}
 & x^2 + 2x + 1 \\
\hline
x^2+x+1 & x^4 + 0\cdot x^3 + x^2 + 0\cdot x + 1 \\
 & \underline{x^4 + x^3 + x^2} \quad \downarrow \quad \downarrow \\
 & 2x^3 + 0\cdot x^2 + 0\cdot x + 1 \\
 & \underline{2x^3 + 2x^2 + 2x} \\
 & x^2 + x + 1 \\
 & \underline{x^2 + x + 1} \\
 & 0
\end{array}
$$

Conclusion / Check:

$$
\begin{aligned}
&(x^2+x+1)(x^2+2x+1) \\
&= x^4 + 2x^3 + x^2 \\
&\qquad + x^3 + 2x^2 + x \\
&\qquad\qquad + x^2 + 2x + 1 \\
&= x^4 + x^2 + 1
\end{aligned}
$$

no remainder

3. $x^4 + x + 1 = \underbrace{(x^2+2)(x^2+1)}_{x^4+2} + x + 2$

$x^2 + 1 = (x+1)(x+2) + 2$

$x+2 = (2x+1)\,2 + 0$

$\Rightarrow \gcd = 2.$

Note: 2 is invertible in $\mathbb{F}_3[x]$ $(2 \cdot 2 = 1)$. So this is the same as gcd = 1.

4. $\boxed{\begin{matrix} x^4 + x + 1 \\ s=1 \\ t=0 \end{matrix}} = (x^2+2)\boxed{\begin{matrix} x^2+1 \\ s=0 \\ t=1 \end{matrix}} + \boxed{\begin{matrix} x+2 \\ s=1 \\ t=2x^2+1 \end{matrix}}$

$\boxed{\begin{matrix} x^2+1 \\ s=0 \\ t=1 \end{matrix}} = (x+1)\boxed{\begin{matrix} x+2 \\ s=1 \\ t=2x^2+1 \end{matrix}} + \boxed{\begin{matrix} 2 \\ s=2x+2 \\ t=x^3+x^2+2x \end{matrix}}$

Check: $(2x+2)(x^4+x+1) + (x^3+x^2+2x)(x^2+1)$

$= 2x^5 + 2x^2 + 2x + 2x^4 + 2x + 2 + x^5 + x^4 + 2x^3 + x^3 + x^2 + 2x = 2$

If (s,t) gives 2, then $(2s, 2t)$ gives 1, i.e. solution is $\boxed{(x+1,\ 2x^3 + 2x^2 + x)}$