Phase Estimation Exercise – Math 4440

Suppose we have a state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle .$$

1. First, determine the possible measurements (with probabilities of each) if we measure in the basis $|0\rangle$, $|1\rangle$.
2. Next, determine the same for the basis $|+\rangle$, $|-\rangle$.
3. For your answers above, show that with trig identities, the two probabilities simplify to $\cos^2(\theta/2)$ and $\sin^2(\theta/2)$. Which is which?

Solution

1. The state we are given is already in the basis $|0\rangle$, $|1\rangle$:

   $$|\Psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{e^{i\theta}}{\sqrt{2}}|1\rangle .$$

   Therefore, we take the magnitudes squared of the amplitudes for the probabilities.

   The amplitude of $|0\rangle$ is
   $$\frac{1}{\sqrt{2}}.$$
   Its magnitude squared is
   $$\left|\frac{1}{\sqrt{2}}\right|^2 = \frac{1}{2}.$$
   (As a real number, magnitude is just absolute value.)

   The amplitude of $|1\rangle$ is
   $$\frac{e^{i\theta}}{\sqrt{2}}.$$
   Its magnitude squared is

   $$\left|\frac{e^{i\theta}}{\sqrt{2}}\right|^2 = \frac{\left|e^{i\theta}\right|^2}{\left|\sqrt{2}\right|^2} = \frac{|\cos\theta + i\sin\theta|^2}{2} = \frac{\cos^2\theta + \sin^2\theta}{2} = \frac{1}{2}.$$

   (Here, we used that $|a+bi|^2 = a^2 + b^2$.)

   Therefore, the probability of measuring $|0\rangle$ is $1/2$ and the probability of measuring $|1\rangle$ is $1/2$. Good thing these add up to 1!

2. In order to answer this question, we need to put the state into the $|+\rangle$, $|-\rangle$ basis. In class, we used change-of-basis to compute

$$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle .$$

By a similar method, we can compute

$$|1\rangle = \frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle .$$

Subsituting these values into our state, we get

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle\right) + \frac{e^{i\theta}}{\sqrt{2}}\left(\frac{1}{\sqrt{2}}|+\rangle - \frac{1}{\sqrt{2}}|-\rangle\right) = \frac{1+e^{i\theta}}{2}|+\rangle + \frac{1-e^{i\theta}}{2}|-\rangle .$$

In this last format, we can read off the amplitudes in order to compute the probabilities.

The amplitude of $|+\rangle$ is

$$\frac{1+e^{i\theta}}{2}.$$

Its magnitude squared is

$$\left|\frac{1+e^{i\theta}}{2}\right|^2 .$$

This is the probability of $|+\rangle$.

Similarly, the probability of $|-\rangle$ is

$$\left|\frac{1-e^{i\theta}}{2}\right|^2 .$$

3. The last question asks us to simplify these magnitudes. We will show how to simplify the first of these. It's basically an exercise in trig identities.

$$\begin{aligned}\left|\frac{1+e^{i\theta}}{2}\right|^2 &= \frac{1}{4}\left|1+\cos\theta + i\sin\theta\right|^2 \\ &= \frac{1}{4}\left((1+\cos\theta)^2 + \sin^2\theta\right) \\ &= \frac{1}{4}\left((1+\cos^2\theta + 2\cos\theta + \sin^2\theta\right) \\ &= \frac{1}{4}((2+2\cos\theta) \\ &= \frac{1}{2}((1+\cos\theta) \\ &= \cos^2(\theta/2)\end{aligned}$$

At this point, we can use the fact that probability must add to 1, or we can do something similar to reduce the other probability to

$$\sin^2(\theta/2).$$