Fourier Exercise – Math 4440

1. Write down the QFT matrix of dimension $8 \times 8$. You can use the 8-th root of unity notation $\omega_8 = e^{i\pi/4}$. But simplify it so that the entries are all from the set \{1, $-1$, $i$, $-i$, $\omega_8$, $-\omega_8$, $i\omega_8$, $-i\omega_8$\}.

Solution.

\[
\begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & \omega & i & i\omega & -1 & -i\omega & -i & -\omega \\
1 & i & -1 & -i & 1 & i & -1 & -i \\
1 & i\omega & -i & \omega & -1 & -\omega & i & -i\omega \\
1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\
1 & -i\omega & i & -\omega & -1 & \omega & -i & i\omega \\
1 & -i & -1 & i & 1 & -i & -1 & i \\
1 & -\omega & -i & -i\omega & -1 & i\omega & i & \omega
\end{pmatrix}
\]

2. The next several questions are a progression.

(a) What do you get if you apply the QFT of size $m = 2^n$ to the state

\[
\frac{1}{\sqrt{m}} \sum_{x=0}^{m-1} |x\rangle
\]

Give the answer in the form

\[
\sum_{x=0}^{m-1} \alpha_x |x\rangle.
\]

where $\alpha_x$ is as explicit as possible.

(b) Now, with reference to the last question, what is the explicit result (compute the coefficients exactly as complex numbers) if $m = 2^2 = 4$?

(c) What about when $m = 8$?

(d) What do you notice? Conjecture what happens in general.

(e) Prove it. (Note, there’s a complex-numbers geometric/computational proof and a two-line linear algebra proof.)

Solution.

(a)

\[
\frac{1}{m} \sum_{y=0}^{m-1} \sum_{k=0}^{m-1} \omega^{ky} |x\rangle.
\]

(b) We are essentially adding up the entries to each row of the matrix, and all but the first vanish. We get

$|0\rangle$. 

(c) Similarly, $|0\rangle$.

(d) It will always be $|0\rangle$.

(e) The two-line proof is that the QFT matrix is invertible. We are asking for $x$ in the equation $Fv = x$, but the inverse of the QFT is its conjugate transpose, $F^\dagger$, so this is $F^\dagger x = v$, i.e. write $v$ as a sum of columns of $F^\dagger$; but the first column is $v$, so the answer is $(1, 0, 0, \ldots, 0)$, i.e. $|0\rangle$.

A more computational proof is to find the sum
\[
\sum_{y=0}^{m-1} \sum_{k=0}^{m/2-1} \omega^{xy}.
\]

Since $\omega$ is a root of unity, if $x$ is non-zero, then this is a sum of the $m$ $m$-th roots of unity, which surround the origin symmetrically and average to 0. (You can use trig to do this explicitly.) But if $x = 0$, then this is a sum of 1’s.

3. The next several questions are a progression.

(a) What do you get if you apply the QFT of size $m = 2^n$ to the state
\[
\frac{1}{\sqrt{m/2}} \sum_{x=0}^{m/2-1} |2x\rangle?
\]

Give the answer in the form
\[
\sum_{x=0}^{m-1} \alpha_x |x\rangle.
\]

where $\alpha_x$ is as explicit as possible.

(b) Now, with reference to the last question, what is the explicit result (compute the coefficients exactly as complex numbers) if $m = 2^2 = 4$?

(c) What about when $m = 8$?

(d) What do you notice? Conjecture what happens in general.

(e) Prove it.

Solution.

(a)
\[
\frac{\sqrt{2}}{m} \sum_{y=0}^{m-1} \sum_{k=0}^{m/2-1} \omega^{2kx} |x\rangle.
\]

(b) We are essentially adding up every second entry in each row of the matrix, and all but the first and second vanish. We get
\[
\frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |2\rangle.
\]
\[ \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |4\rangle. \]

(d) It will always be \( \frac{1}{\sqrt{2}} |0\rangle + \frac{1}{\sqrt{2}} |4\rangle. \)

(e) We are asking for \( x \) in the equation \( Fv = x \), but the inverse of the QFT is its conjugate transpose, \( F^\dag \), so this is \( F^\dag x = v \), i.e. write \( v \) as a sum of columns of \( F^\dag \); but the first column plus second column add to \( v \), so the answer is as conjectured.

A more computational proof is to find the sum

\[
\sum_{y=0}^{m-1} \sum_{k=0}^{m/2-1} \omega^{2ky}. 
\]

Since \( \omega \) is a root of unity, if \( x \) is non-zero mod \( m \), then this is a sum of the \( m \) \( m \)-th roots of unity, which surround the origin symmetrically and average to 0. (You can use trig to do this explicitly.) But if \( x = 0 \) or \( m/2 \), then this is a sum of 1’s.