Continued Fraction and Intro Coding Exercises – Math 4440

Continued Fraction

Compute the continued fraction of 31/64:

\[
\frac{31}{64} = \frac{1}{\frac{64}{31}} = \frac{1}{2 + \frac{2}{\frac{31}{2}}} = \frac{1}{2 + \frac{1}{\frac{15}{2}}} = \frac{1}{2 + \frac{1}{15 + \frac{1}{2}}}
\]

At this point, we know we are done because all numerators are 1.

Coding Theory

Consider the code \(C = \{(0, 0, 1), (1, 1, 1), (1, 0, 0), (0, 1, 0)\}\).

1. What is the length of \(C\)?
2. What is \(d(C)\) (the minimum distance)?
3. What are \(q, n, M\) and \(d\) if \(C\) is a \(q\)-ary \((n, M, d)\)-code?
4. What is the code rate of \(C\)?
5. How many errors can \(C\) detect?
6. How many errors can \(C\) correct?
7. Show that \(C\) is not linear.
8. Suppose you send the codeword \((1, 1, 1)\) and 2 errors are made on the noisy channel, in the first and last positions. Explain what message is received and what it decodes to. Was communication successful?
9. Show that \(C\) is a coset of a linear code. What is the linear code? Call it \(C'\).
10. Find \(n\) and \(k\) so that \(C'\) is an \([n, k]\)-code.
11. Define \(C'\) by linear equation(s).
12. Give a basis for \(C'\).
13. Give a matrix for which \(C'\) is the rowspace.

Solutions.

1. The length is 3 since there are 3 characters in each codeword.
2. We have to find the smallest distance between codewords. There are 6 pairs to check, and among these, we find that all pairs differ by 2. So \(d(C) = 2\).
3. $C$ is a 2-ary (binary) $(3, 4, 2)$-code.

4. The code rate is 
\[ R = \frac{\log_2(4)}{3} = \frac{2}{3}. \]

5. $C$ can detect 1 error.

6. $C$ can correct 0 errors.

7. To show $C$ is not linear, we can find two codewords whose sum is not a codeword. For example,
\[ (0, 0, 1) + (1, 1, 1) = (1, 1, 0) \notin C. \]
Alternatively, we can notice that $(0, 0, 0)$ is not in $C$, so it cannot be a subspace.

8. Suppose you send the codeword $(1, 1, 1)$ and 2 errors are made on the noisy channel, in the first and last positions. Then the message that is received is $(0, 1, 0)$, since first and last positions are "flipped". This message is a codeword, so it decodes to itself. The communication was not successful, since the decoded message isn’t the message that was sent.

9. If we subtract $(0, 0, 1)$ from every element of $C$, we get
\[ C' = \{(0, 0, 0), (1, 1, 0), (1, 0, 1), (0, 1, 1)\}. \]
Therefore $C = (0, 0, 1) + C'$. The code $C'$ is linear because it is a subspace of $\mathbb{F}_3^2$ (in other words, it contains 0 and is closed under addition; there are few enough elements to check exhaustively!).

10. $n = 3$ and $k = 2$. One way to see this is to give a basis for $C'$ which we do below. Another is to count: a $k$-dimensional subspace is of cardinality $2^k$. $C'$ it has 4 elements, so it is of cardinality $2^k$ for $k = 2$.

11. We expect one linear equation since it is of codimension 1 in the ambient vector space. The equation $x + y + z = 0$ seems to do it.

12. A basis is $(1, 1, 0), (1, 0, 1)$. These are linearly independent. But actually any two non-zero vectors of $C'$ form a basis, so your answer may differ.

13. Just put the basis elements into a matrix as the rows:
\[
\begin{pmatrix}
1 & 1 & 0 \\
1 & 0 & 1
\end{pmatrix}
\]