2 qubits

classical states: 00 01 10 11

quantum superposition: $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$.

w/ $\alpha_{ij} \in \mathbb{C}$, $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$.

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w/ $\alpha_{ij} \in \mathbb{C}$, $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$.

Measure Both Qubits:

Result	Probability	State after measurement
00	$\lvert\alpha_{00}\rvert^2$	$\lvert 00\rangle$
01	$\lvert\alpha_{01}\rvert^2$	$\lvert 01\rangle$
10	$\lvert\alpha_{10}\rvert^2$	$\lvert 10\rangle$
11	$\lvert\alpha_{11}\rvert^2$	$\lvert 11\rangle$

2 qubits

classical states: 00 01 10 11

quantum superposition: $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$.

w/ $\alpha_{ij} \in \mathbb{C}$, $|\alpha_{00}|^2 + |\alpha_{01}|^2 + |\alpha_{10}|^2 + |\alpha_{11}|^2 = 1$.

Measure 1st Qubit:

Result	Probability	State after measurement
0	$\lvert\alpha_{00}\rvert^2 + \lvert\alpha_{01}\rvert^2$	$\dfrac{\alpha_{00}\lvert 00\rangle + \alpha_{01}\lvert 01\rangle}{\sqrt{\lvert\alpha_{00}\rvert^2 + \lvert\alpha_{01}\rvert^2}}$
1	$\lvert\alpha_{10}\rvert^2 + \lvert\alpha_{11}\rvert^2$	$\dfrac{\alpha_{10}\lvert 10\rangle + \alpha_{11}\lvert 11\rangle}{\sqrt{\lvert\alpha_{10}\rvert^2 + \lvert\alpha_{11}\rvert^2}}$

Measure 2nd Qubit:

First Qubit	Result (2nd)	Probability	x Prob of 1st msr
0	0	$\dfrac{\lvert\alpha_{00}\rvert^2}{\cancel{\left(\sqrt{\lvert\alpha_{00}\rvert^2 + \lvert\alpha_{01}\rvert^2}\right)^2}}$	$\cancel{(\lvert\alpha_{00}\rvert^2 + \lvert\alpha_{01}\rvert^2)} = \lvert\alpha_{00}\rvert^2$
	1		
1	0		
	1		

2 qubits

classical states:

quantum superposition:

Measure 1st Qubit:

Result	Probability	State after measurement
0	$\lvert\alpha_{00}\rvert^2 + \lvert\alpha_{01}\rvert^2$	$\left(\alpha_{00}\lvert 00\rangle + \alpha_{01}\lvert 01\rangle\right)\left(\sqrt{\lvert\alpha_{00}\rvert^2 + \lvert\alpha_{01}\rvert^2}\right)^{-1}$
1	$\lvert\alpha_{10}\rvert^2 + \lvert\alpha_{11}\rvert^2$	$\left(\alpha_{10}\lvert 10\rangle + \alpha_{11}\lvert 11\rangle\right)\left(\sqrt{\lvert\alpha_{10}\rvert^2 + \lvert\alpha_{11}\rvert^2}\right)^{-1}$

Measure 2nd Qubit:

First Qubit	Result (2nd)	Probability

2 qubits

classical states:

quantum superposition:

Measure 1st Qubit:

Result	Probability	State after measurement
0	$\lvert\alpha_{00}\rvert^2 + \lvert\alpha_{01}\rvert^2$	$(\alpha_{00}\lvert 00\rangle + \alpha_{01}\lvert 01\rangle)\left(\sqrt{\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2}\right)^{-1}$
1	$\lvert\alpha_{10}\rvert^2 + \lvert\alpha_{11}\rvert^2$	$(\alpha_{10}\lvert 10\rangle + \alpha_{11}\lvert 11\rangle)\left(\sqrt{\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2}\right)^{-1}$

Measure 2nd Qubit:

First Qubit	Result (2nd)	Probability
0	0	$\lvert\alpha_{00}\rvert^2 \left(\sqrt{\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2}\right)^{-2}\left(\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2\right)$
	1	$\lvert\alpha_{01}\rvert^2 \left(\sqrt{\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2}\right)^{-2}\left(\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2\right)$
1	0	$\lvert\alpha_{10}\rvert^2 \left(\sqrt{\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2}\right)^{-2}\left(\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2\right)$
	1	$\lvert\alpha_{11}\rvert^2 \left(\sqrt{\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2}\right)^{-2}\left(\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2\right)$

2 qubits

classical states:

quantum superposition:

Measure 1^{st} Qubit:

Result	Probability	State after measurement
0	$\lvert\alpha_{00}\rvert^2 + \lvert\alpha_{01}\rvert^2$	$(\alpha_{00}\lvert 00\rangle + \alpha_{01}\lvert 01\rangle)\left(\sqrt{\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2}\right)^{-1}$
1	$\lvert\alpha_{10}\rvert^2 + \lvert\alpha_{11}\rvert^2$	$(\alpha_{10}\lvert 10\rangle + \alpha_{11}\lvert 11\rangle)\left(\sqrt{\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2}\right)^{-1}$

Measure 2^{nd} Qubit:

First Qubit	Result (2^{nd})	Probability
0	0	$\lvert\alpha_{00}\rvert^2 \cancel{\left(\sqrt{\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2}\right)^{-2}\left(\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2\right)}$
	1	$\lvert\alpha_{01}\rvert^2 \cancel{\left(\sqrt{\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2}\right)^{-2}\left(\lvert\alpha_{00}\rvert^2+\lvert\alpha_{01}\rvert^2\right)}$
1	0	$\lvert\alpha_{10}\rvert^2 \cancel{\left(\sqrt{\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2}\right)^{-2}\left(\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2\right)}$
	1	$\lvert\alpha_{11}\rvert^2 \cancel{\left(\sqrt{\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2}\right)^{-2}\left(\lvert\alpha_{10}\rvert^2+\lvert\alpha_{11}\rvert^2\right)}$

It doesn't matter what order we measure in!

Unentangled qubits:

1st: $|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle$

2nd: $|\psi_2\rangle = a_2|0\rangle + b_2|1\rangle$

joint state: $a_1a_2|00\rangle + a_1b_2|01\rangle + a_2b_1|10\rangle + b_1b_2|11\rangle$

Measure 1st qubit:

outcome	probability	resulting state
0	$\|a_1a_2\|^2 + \|a_1b_2\|^2$ $= \|a_1\|^2(\|a_2\|^2 + \|b_2\|^2)$ $= \|a_1\|^2$	$\dfrac{a_1a_2\|00\rangle + a_1b_2\|01\rangle}{\sqrt{\|a_1a_2\|^2 + \|a_1b_2\|^2}}$ $= \dfrac{a_1}{\sqrt{\|a_1\|^2}}\left(a_2\|00\rangle + b_2\|01\rangle\right)$ $\sim a_2\|00\rangle + b_2\|01\rangle$

Unentangled qubits:

1st: $|\psi_1\rangle = a_1|0\rangle + b_1|1\rangle$

2nd: $|\psi_2\rangle = a_2|0\rangle + b_2|1\rangle$

joint state: $a_1 a_2|00\rangle + a_1 b_2|01\rangle + a_2 b_1|01\rangle + b_1 b_2|11\rangle$

Measure 1st qubit:

outcome	probability	resulting state
0	$\|a_1\|^2$	$a_2|00\rangle + b_2|01\rangle$
1	$\|b_1\|^2$	$a_2|10\rangle + b_2|11\rangle$

Bottom Line: they behave as unrelated systems.

The joint state has a name: the <u>tensor product</u> of $|\psi_1\rangle$ and $|\psi_2\rangle$, denoted $|\psi_1\rangle \otimes |\psi_2\rangle$.

Entangled qubits

joint state: $\alpha_{00}|00\rangle + \alpha_{01}|01\rangle + \alpha_{10}|10\rangle + \alpha_{11}|11\rangle$

which is not a tensor product of 2 separate states.

Example: $\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$ entangled.

Is this a tensor product? NO

$$\left.\begin{aligned} \tfrac{1}{\sqrt{2}} &= a_1 a_2 \\ 0 &= a_1 b_2 \\ 0 &= b_1 a_2 \\ \tfrac{1}{\sqrt{2}} &= b_1 b_2 \end{aligned}\right\} \text{no solution in } a_1, a_2, b_1, b_2$$

Spooky Action at a distance:

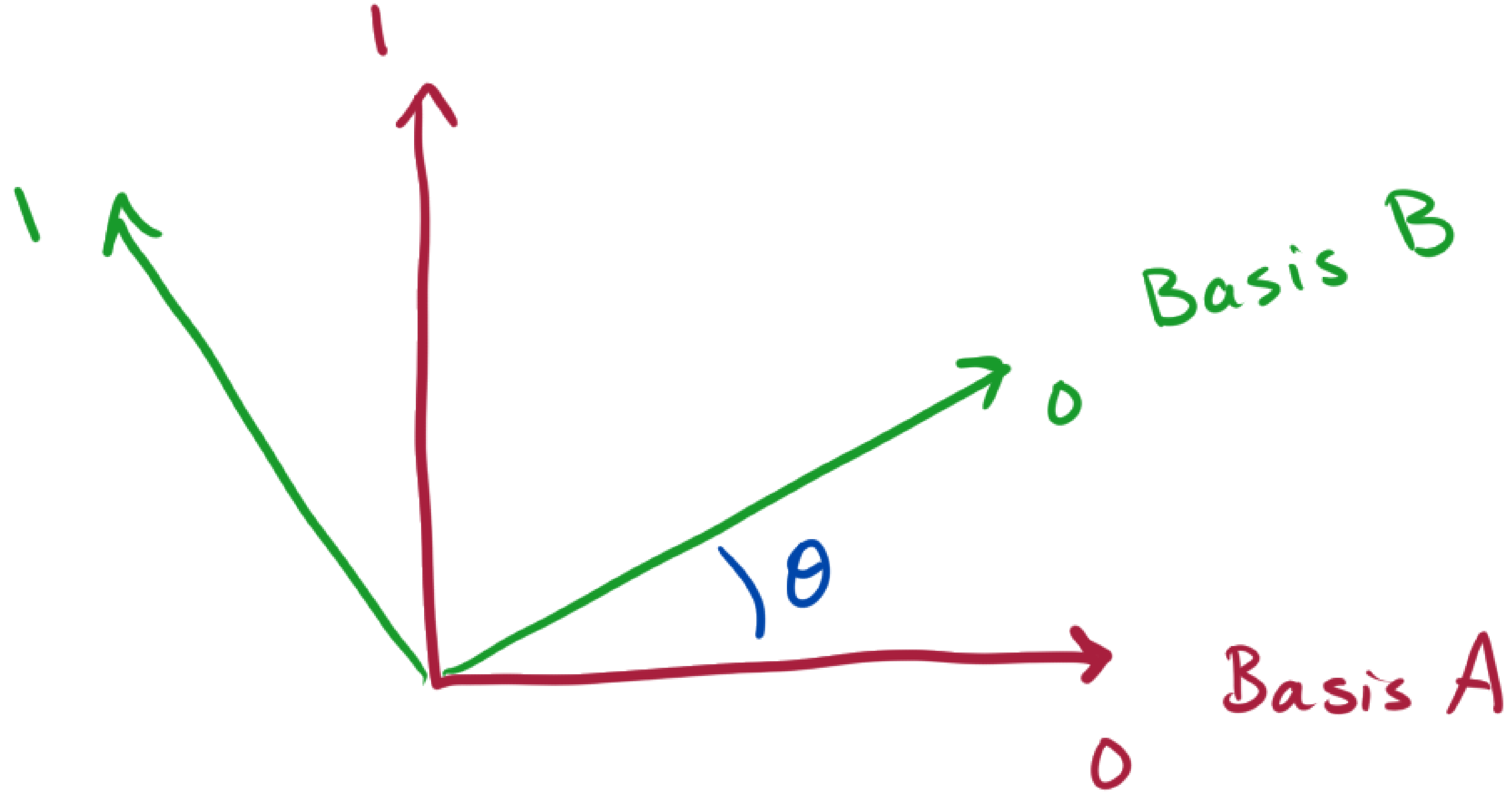
"Bell state": $|\psi\rangle = \frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle.$

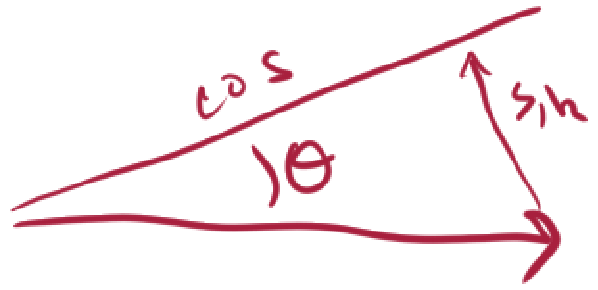
(exercise)

$= \frac{1}{\sqrt{2}}|++\rangle + \frac{1}{\sqrt{2}}|--\rangle$

$= \frac{1}{\sqrt{2}}|e_1 e_1\rangle + \frac{1}{\sqrt{2}}|e_2 e_2\rangle$ for any orthogonal basis e_1, e_2.

Consider two bases:





Alice measures Bell state 1st qubit:

Basis A
$|0\rangle, |1\rangle$

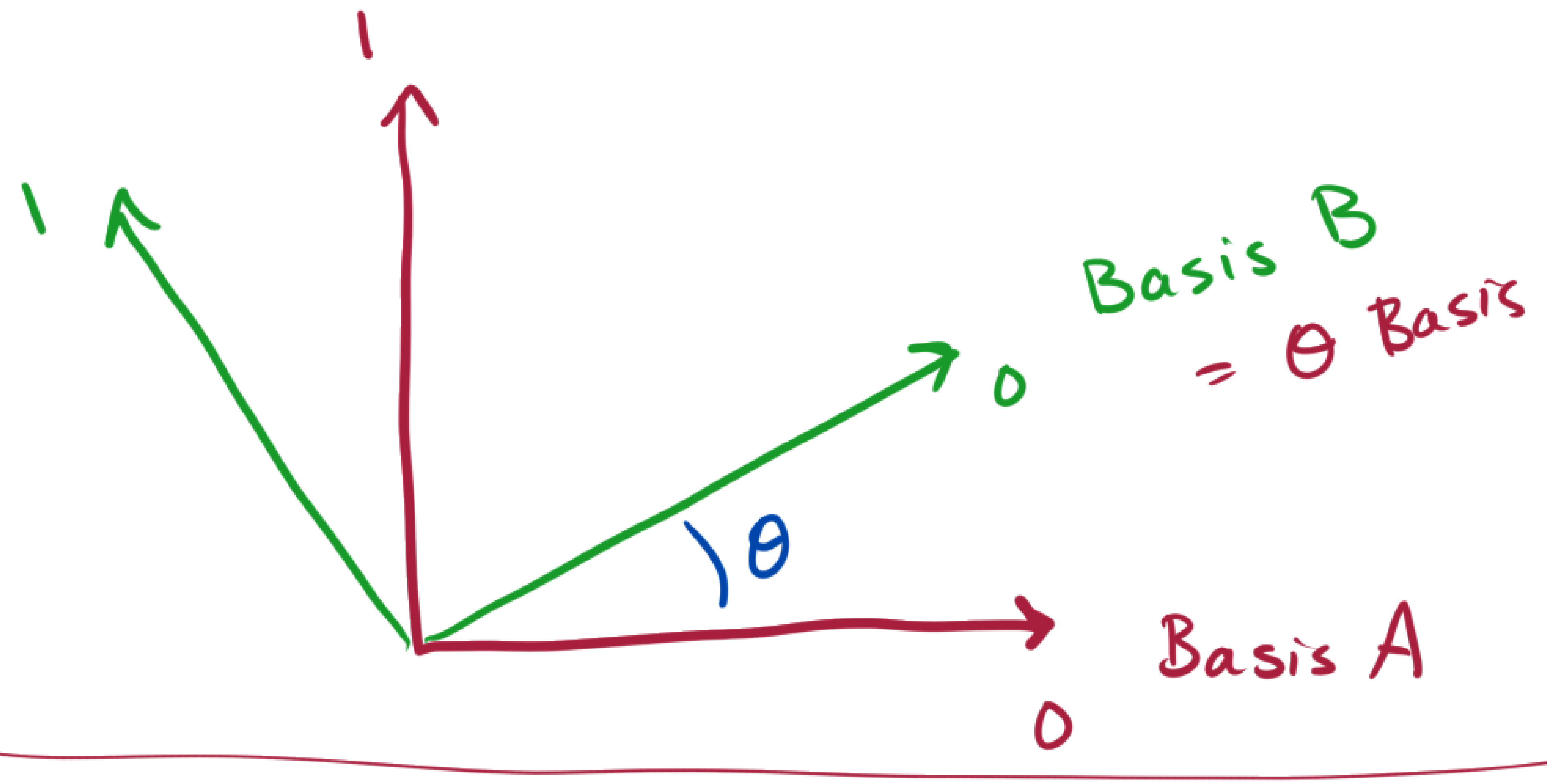
Outcome	Prob	Resulting State
0	1/2	$\|00\rangle$
1	1/2	$\|11\rangle$

Bob measures Bell state 2nd qubit:

Bases B

Alice's Outcome	Bob's	Probability	if $\theta = 0$
$\|00\rangle$	0	$\cos^2\theta$	1
	1	$\sin^2\theta$	0
$\|11\rangle$	0	$\sin^2\theta$	0
	1	$\cos^2\theta$	1

Consider two bases:



Final Table:

A, B measurement	Probability
0, 0	$\frac{\cos^2\theta}{2}$
0, 1	$\frac{\sin^2\theta}{2}$
1, 0	$\frac{\sin^2\theta}{2}$
1, 1	$\frac{\cos^2\theta}{2}$

Alice measures Bell state 1st qubit:

Basis A
$|0\rangle, |1\rangle$

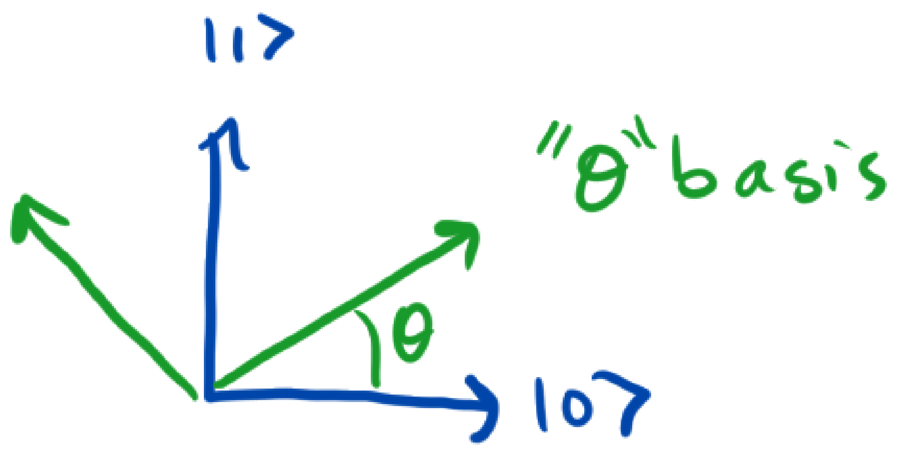
Outcome	Prob	Resulting State
0	1/2	$\|00\rangle$
1	1/2	$\|11\rangle$

Bob measures Bell state 2nd qubit:

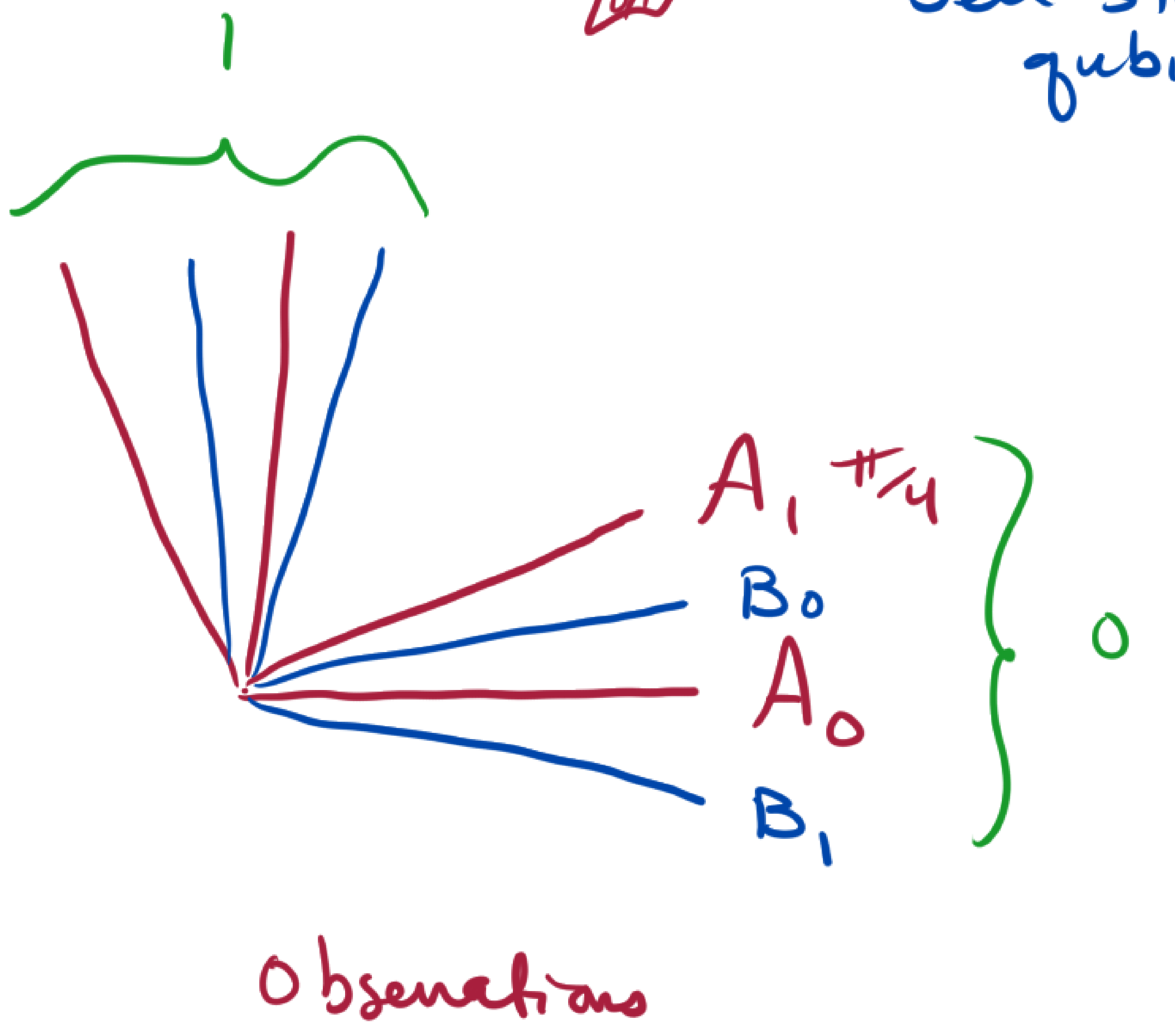
Bases B

Alice's Outcome	Bob's	Probability	if $\theta=0$
$\|00\rangle$	0	$\cos^2\theta$	1
	1	$\sin^2\theta$	0
$\|11\rangle$	0	$\sin^2\theta$	0
	1	$\cos^2\theta$	1

EPR Paradox / Bell Test — can quantum mechanics just be hidden variables?



Experiment:



① Generate random bits (classical) X_A for Alice, X_B for Bob

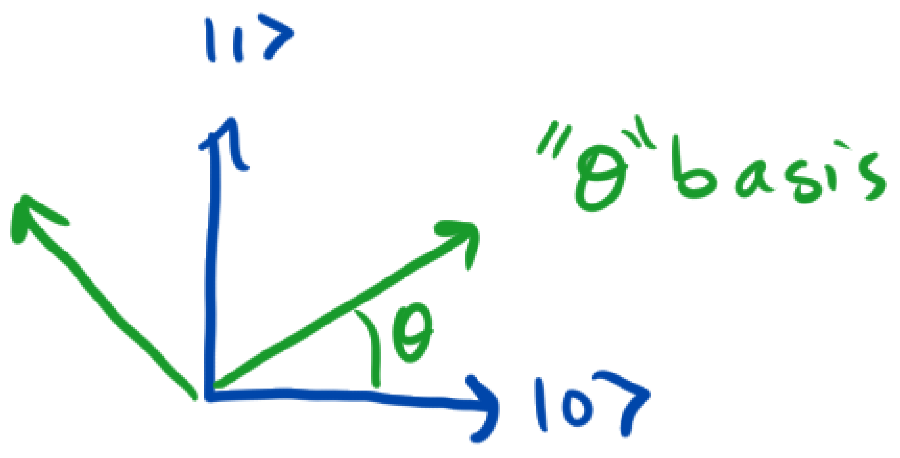
② **Alice:** If $X_A = 0$ measures in $|0\rangle, |1\rangle$ basis
If $X_A = 1$ " " $\pi/4$ basis
} O_A

Bob: If $X_B = 0$ measures in $\pi/8$ basis
If $X_B = 1$ " " $-\pi/8$ basis
} O_B

③ Compute the statistic:

$$\text{Prob}\left(X_A X_B \pmod 2 \equiv O_A + O_B \pmod 2 \right)$$

EPR Paradox / Bell Test — can quantum mechanics just be hidden variables?



Statistic: $\mathrm{Prob}\Big(\overbrace{X_A X_B}^{\text{random bits}} \equiv \overbrace{O_A + O_B}^{\text{measurements}} \pmod 2 \Big)$

$= \mathrm{Prob}\left(\begin{array}{c} \text{at least one bit is 0 and measurements agree} \\ \text{OR} \\ \text{both bits are 1 and measurements disagree} \end{array} \right)$

Analysis: $X_A X_B \equiv 0 \iff$ Alice & Bob's bases differ by $\frac{\pi}{8}$.

$\iff$ Probability that outputs agree is $\cos^2(\frac{\pi}{8})$.

$X_A X_B \equiv 1 \iff$ bases differ by $\frac{3\pi}{8}$.

$\iff$ Probability that outputs disagree is $\sin^2(\frac{3\pi}{8}) = \cos^2(\frac{\pi}{8})$.

So total probability that Statistic = TRUE is $\cos^2(\frac{\pi}{8}) = \frac{2+\sqrt{2}}{4} \doteq 0.85$.

Best Hidden Variable Strategy:

Alice's Strategy

X_A	Prob of measuring 0	or 1
0	p	$1-p$
1	q	$1-q$

Bob's Strategy

X_B	Prob of measuring 0	or 1
0	r	$1-r$
1	s	$1-s$

Best Hidden Variable Strategy:

Alice's Strategy

X_A	Prob of measuring 0	or 1
0	p	$1-p$
1	q	$1-q$

Bob's Strategy

X_B	Prob of measuring 0	or 1
0	r	$1-r$
1	s	$1-s$

Best Hidden Variable Strategy:

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0	p	$1-p$
1	q	$1-q$

Bob's Strategy

X_B	Prob of measuring 0	or 1
0	r	$1-r$
1	s	$1-s$

$$\text{Prob}(X_A X_B \equiv O_A + O_B)$$

$$= \tfrac{1}{4}\text{Prob}(O_A = O_B \mid X_A = X_B = 0) + \tfrac{1}{4}\text{Prob}(O_A = O_B \mid X_A = 1, X_B = 0) + \tfrac{1}{4}\text{Pr}(O_A = O_B \mid X_A = 0, X_B = 1)$$
$$+ \tfrac{1}{4}\text{Pr}(O_A \neq O_B \mid X_A = X_B = 1)$$

$$= \tfrac{1}{4}\left[pr + (1-p)(1-r) + ps + (1-p)(1-s) + rq + (1-r)(1-q) + q(1-s) + s(1-q)\right]$$

$$= \tfrac{1}{4}\left[2(pr + ps + rq - sq) + 3 - 2(p+r)\right]$$

This is $\leq \frac{3}{4}$ (reached when $p=q=r=s=0$ or $p=q=r=s=1$).

Best Hidden Variable Strategy:

Alice's Strategy

X_A	Prob of measuring 0	or 1
0	p	$1-p$
1	q	$1-q$

Bob's Strategy

X_B	Prob of measuring 0	or 1
0	r	$1-r$
1	s	$1-s$

$$\mathrm{Prob}(X_A X_B \equiv O_A + O_B)$$

$$= \frac{1}{4}\mathrm{Prob}(O_A = O_B \mid X_A = X_B = 0) + \frac{1}{4}\mathrm{Prob}(O_A = O_B \mid X_A = 1, X_B = 0) + \frac{1}{4}\Pr(O_A = O_B \mid X_A = 0, X_B = 1)$$
$$+ \frac{1}{4}\Pr(O_A \neq O_B \mid X_A = X_B = 1)$$

$$= \frac{1}{4}\left[pr + (1-p)(1-r) + ps + (1-p)(1-s) + rq + (1-r)(1-q) + q(1-s) + s(1-q) \right]$$

$$= \frac{1}{4}\left[2(pr + ps + rq - sq) + 3 - 2(p+r) \right]$$

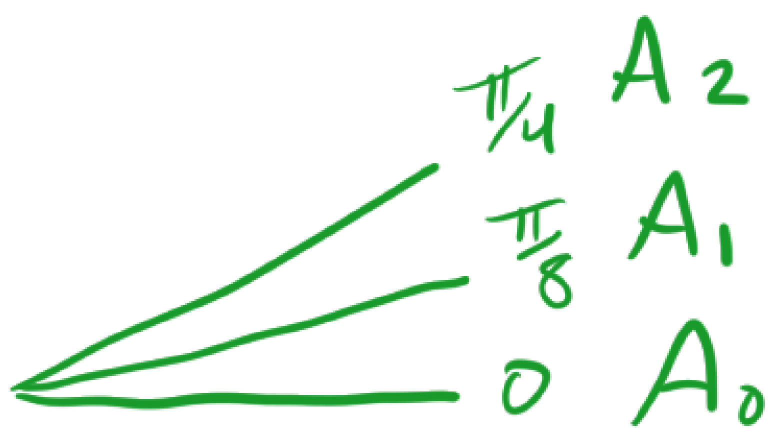
This is $\leq \frac{3}{4}$ (reached when $p = q = r = s = 0$ or $p = q = r = s = 1$).

Quantum Key Distribution with Entanglement (Ekert '91)

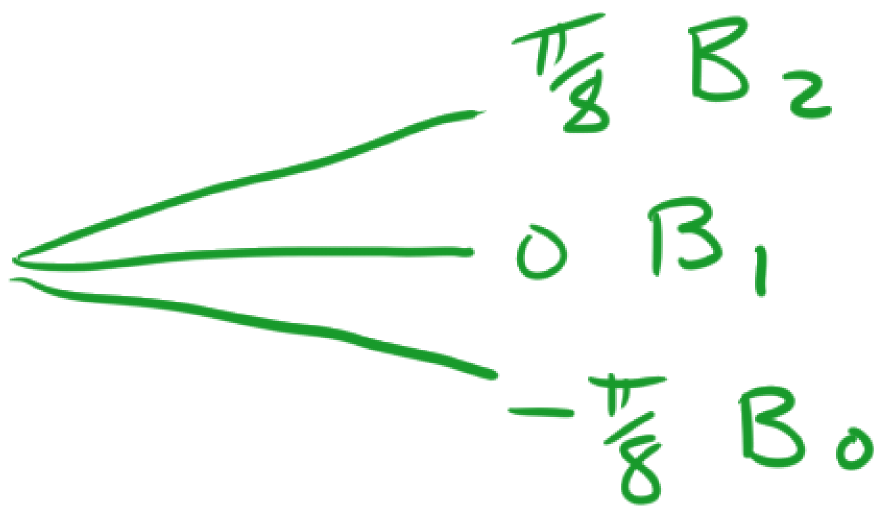
Alice $\longleftarrow$ Bell state $\longrightarrow$ Bob

$$\frac{1}{\sqrt{2}}|00\rangle + \frac{1}{\sqrt{2}}|11\rangle$$

Bases:



Bases:



Both measure with random bases:

Prob $= \frac{2}{9}$ they agree (A_0 with B_1 or A_1 with B_2)

$\Longrightarrow$ derive key

Remaining data

$\Longrightarrow$ Bell test to detect Eve