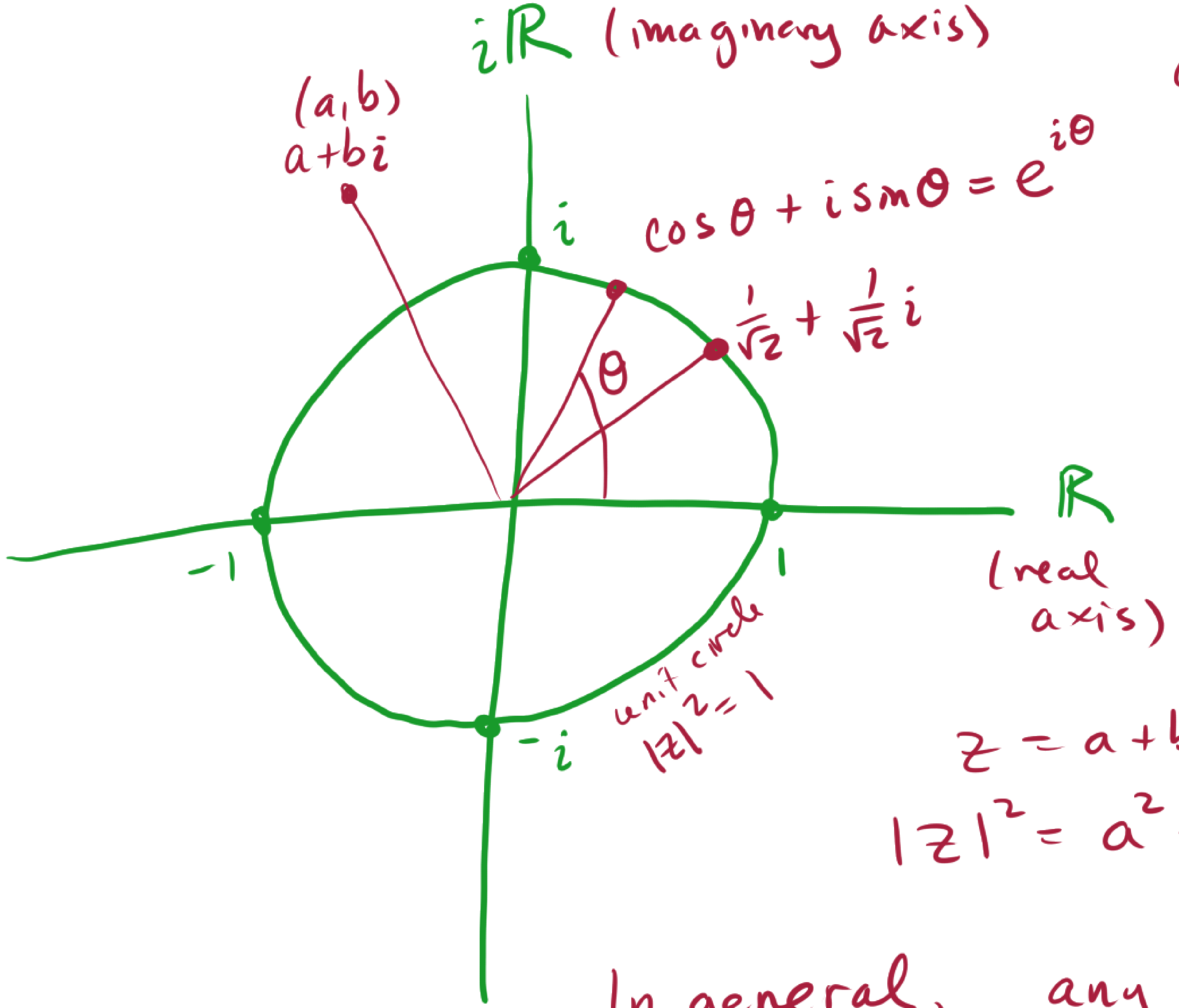
Complex Numbers

$\mathbb{C} = \mathbb{R} + i\mathbb{R} = \{a + bi : a, b \in \mathbb{R}\}$ a field



add: $(a+bi) + (c+di) = (a+c) + (b+d)i$

(addition of vectors)

multiply: as normal except $i^2 = -1$.

e.g. $(2+3i)(5+7i)$

$= 2\cdot 5 + 3\cdot 5 i + 2\cdot 7 i + 3\cdot 7 i^2$

$= \underbrace{(2\cdot 5 - 3\cdot 7)}_{\text{real part}} + \underbrace{(3\cdot 5 + 2\cdot 7)}_{\text{imaginary part}} i$

$z = a + bi$

$|z|^2 = a^2 + b^2$

In general, any complex number can be expressed as $z = \underset{\text{length } |z|}{r}\, e^{i\theta}$ ← angle to $\mathbb{R}$ axis

Multiplication: $r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1 + \theta_2)}$

Quantum Mechanics: The qubit

e.g. electrons $\begin{pmatrix} \text{spin up} = 0 \\ \text{spin down} = 1 \end{pmatrix}$

photons $\begin{pmatrix} \text{horizontal} = 0 \\ \text{vertical} = 1 \end{pmatrix}$

Mathematically, a qubit is in a state which is a complex linear combination of the classical '0' and '1' states.

Classical states: $|0\rangle$, $|1\rangle$

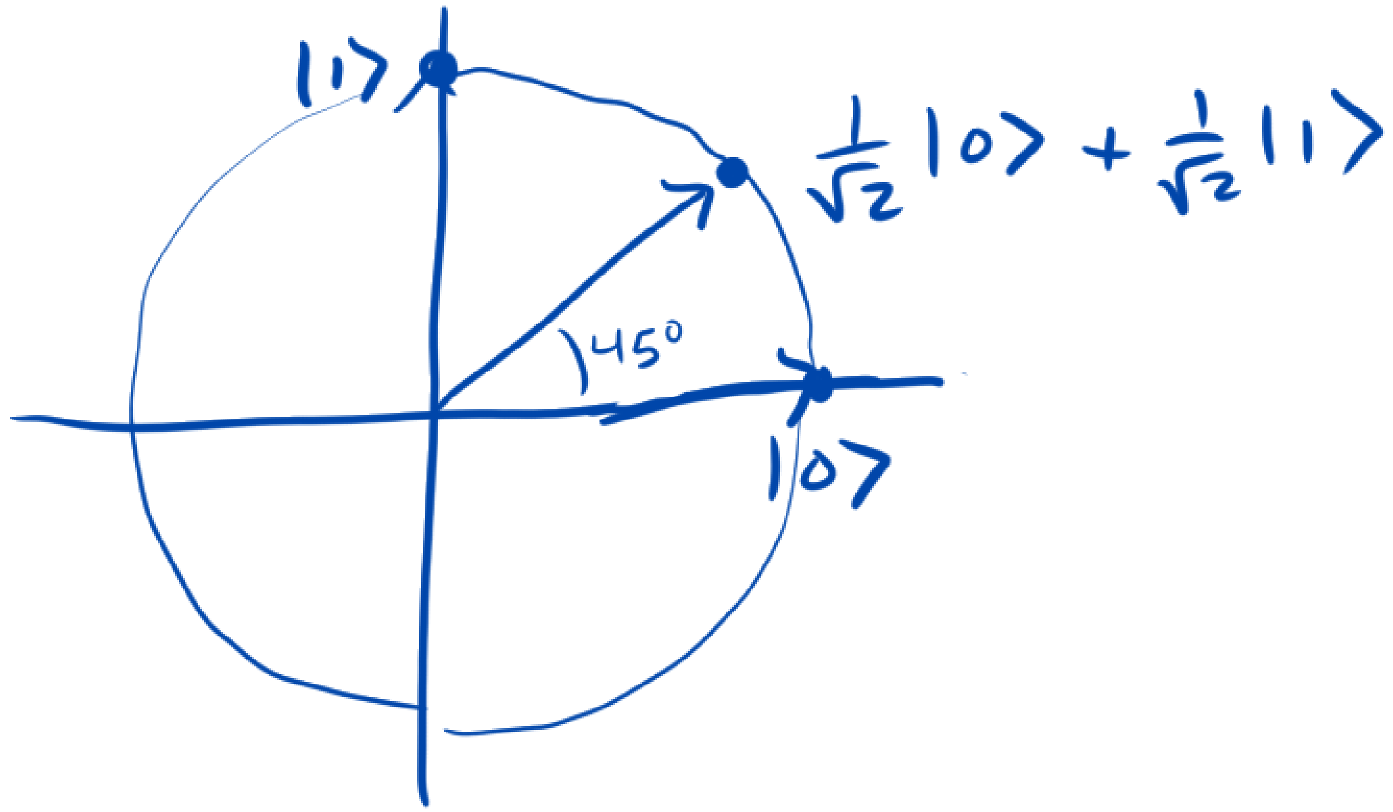
Qubit states (superposition):

The state of a qubit is something of the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$$

where $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 = 1$.

This is a superposition of $|0\rangle$ and $|1\rangle$ with amplitudes α and β.



Think of $|0\rangle$, $|1\rangle$ as vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$.

and superpositions as length-1 vectors in the complex vectorspace spanned by $|0\rangle$ and $|1\rangle$.

Hilbert space.

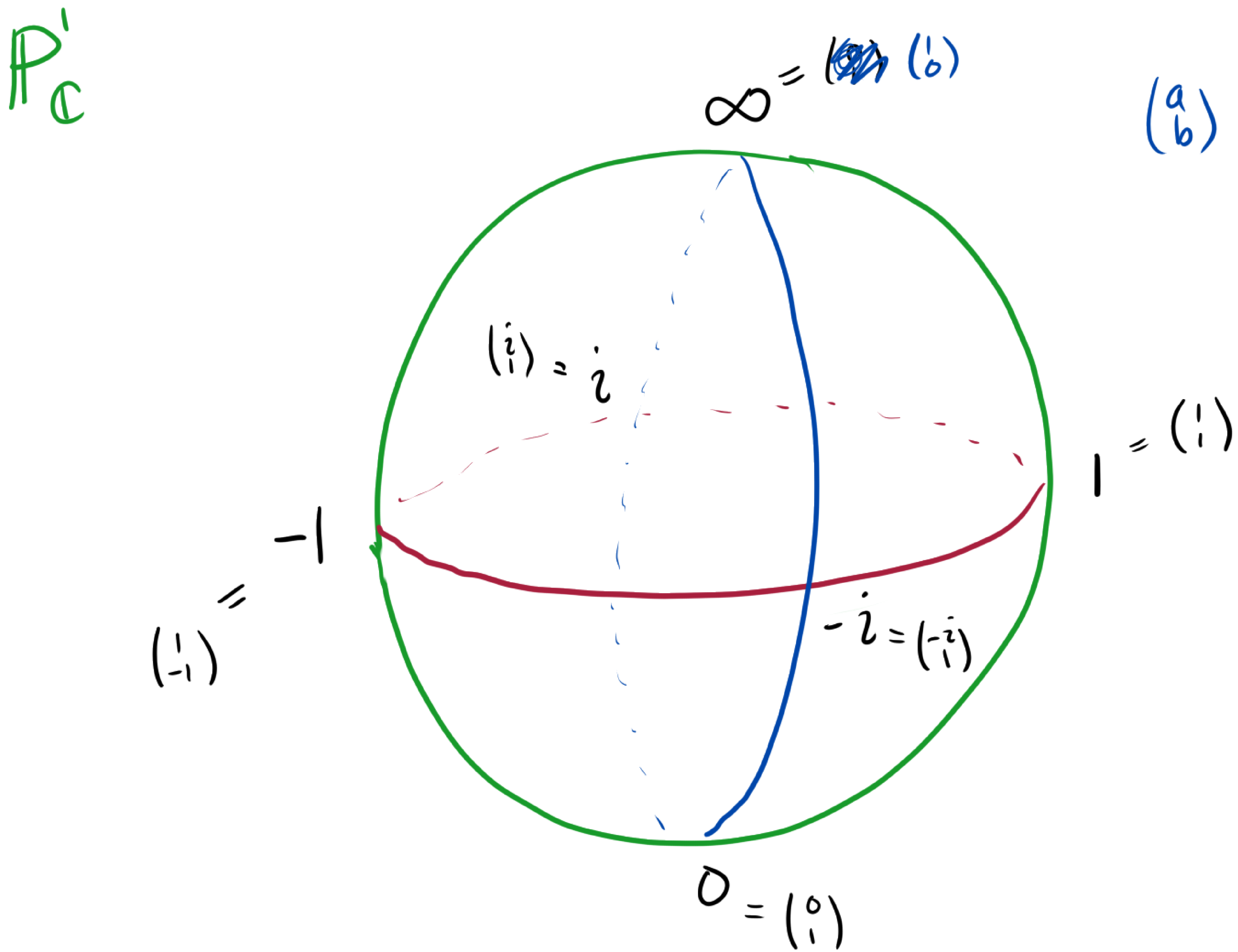
e.g. $\frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$

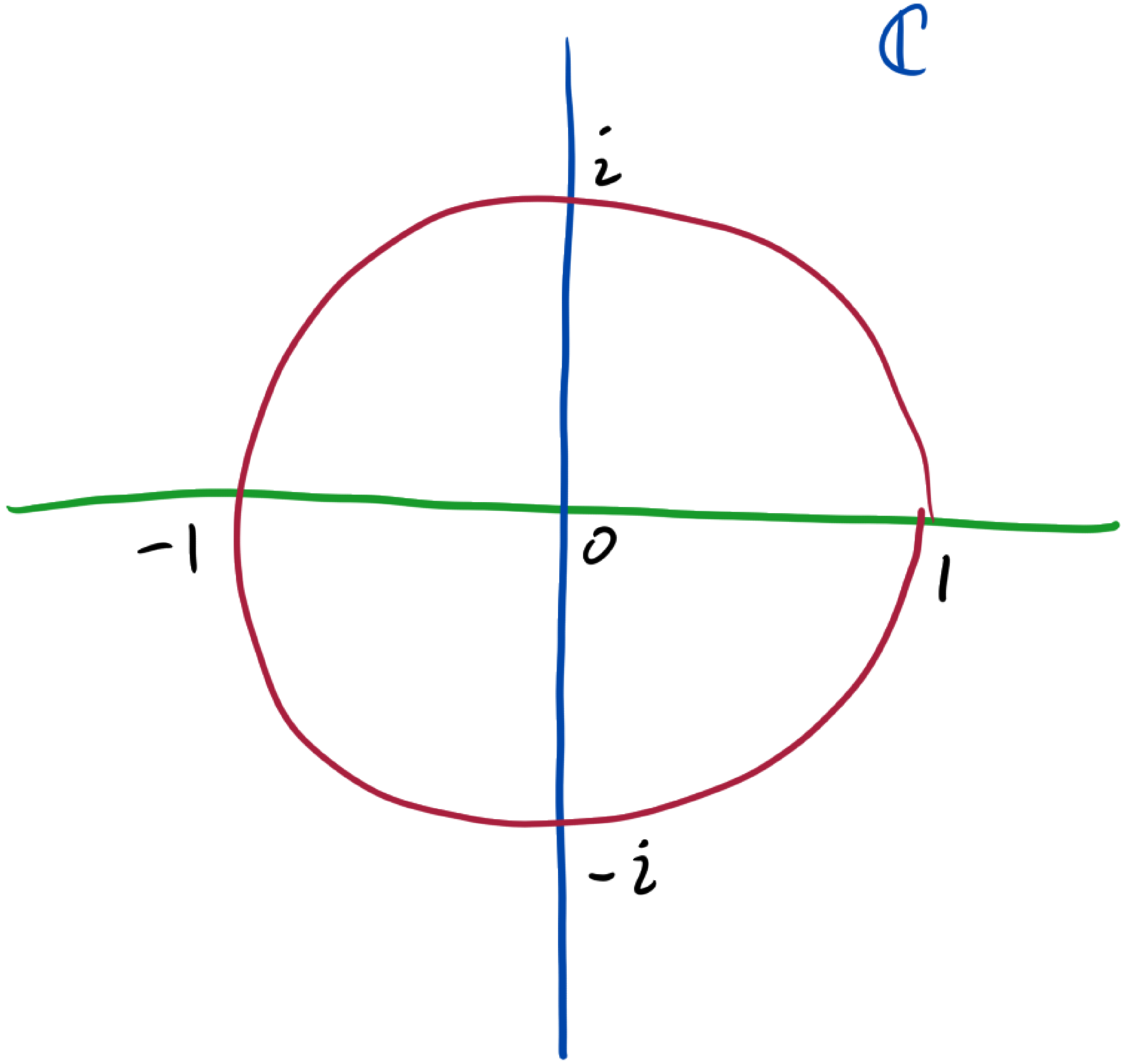
Notes: (1) Normalizing any vector $v = a|0\rangle + b|1\rangle$ to length one $\left(|a|^2 + |b|^2 = 1 \right)$ means scaling by $\underset{(\text{real})}{|v|}$: $\frac{a|0\rangle + b|1\rangle}{|v|}$

(2) The states $a|0\rangle + b|1\rangle$ and $\underset{\uparrow \text{ length } 1}{e^{i\theta}}(a|0\rangle + b|1\rangle)$ are indistinguishable.

$\Longrightarrow$ State space is the space of all slopes $\frac{a}{b}$ for $a|0\rangle + b|1\rangle$.

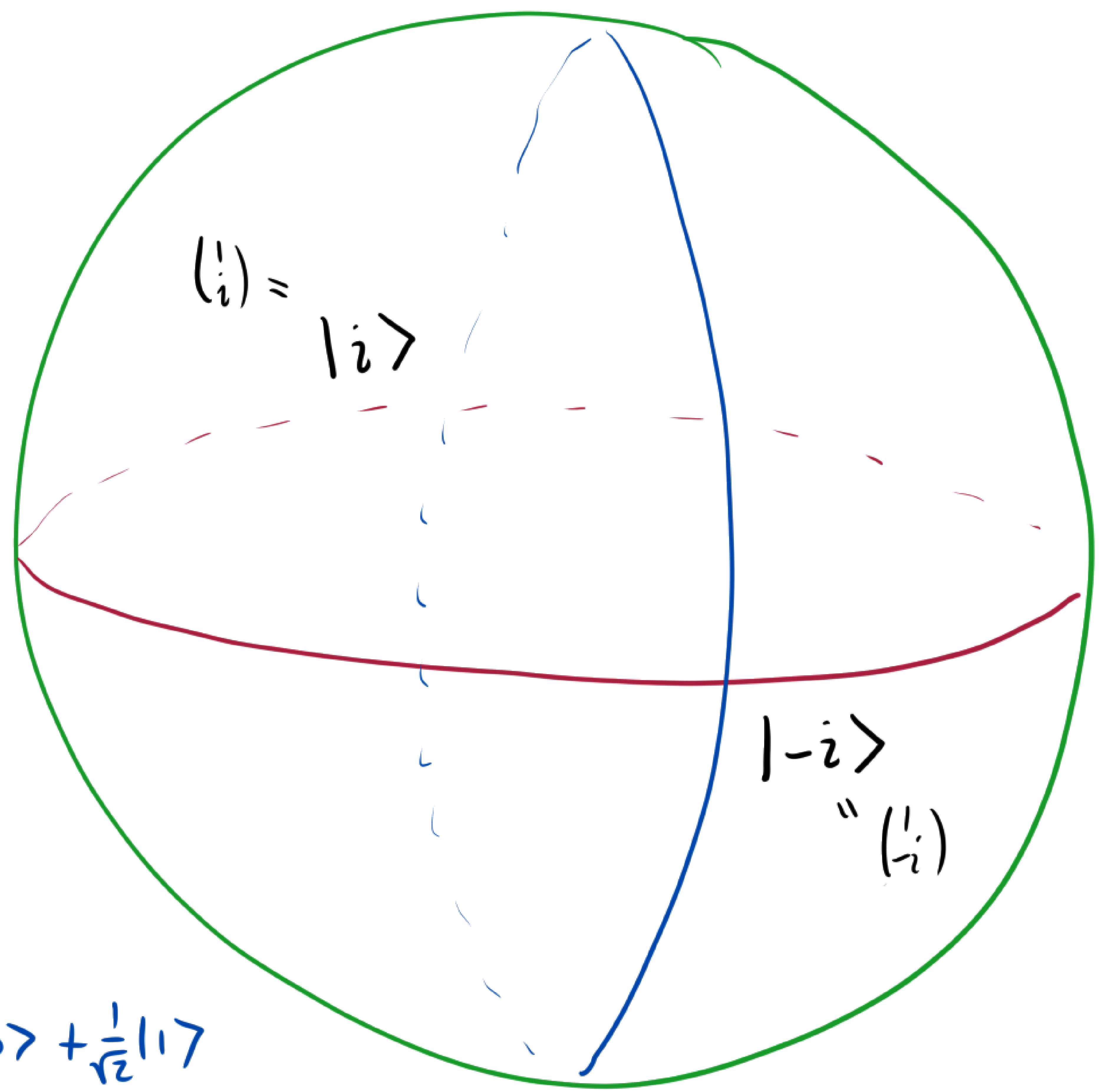
$$= \mathbb{C} \cup \{\infty\}$$


$\mathbb{P}^1_{\mathbb{C}}$
$\infty = \begin{pmatrix}1\\0\end{pmatrix}$
$\begin{pmatrix}a\\b\end{pmatrix}$
$\begin{pmatrix}i\\1\end{pmatrix} = i$
$1 = \begin{pmatrix}1\\1\end{pmatrix}$
$-1 = \begin{pmatrix}1\\-1\end{pmatrix}$
$-i = \begin{pmatrix}-i\\1\end{pmatrix}$
$0 = \begin{pmatrix}0\\1\end{pmatrix}$


$\mathbb{C}$
i
-1
0
1
$-i$

Bloch Sphere

(state space of one qubit)



$$|+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$

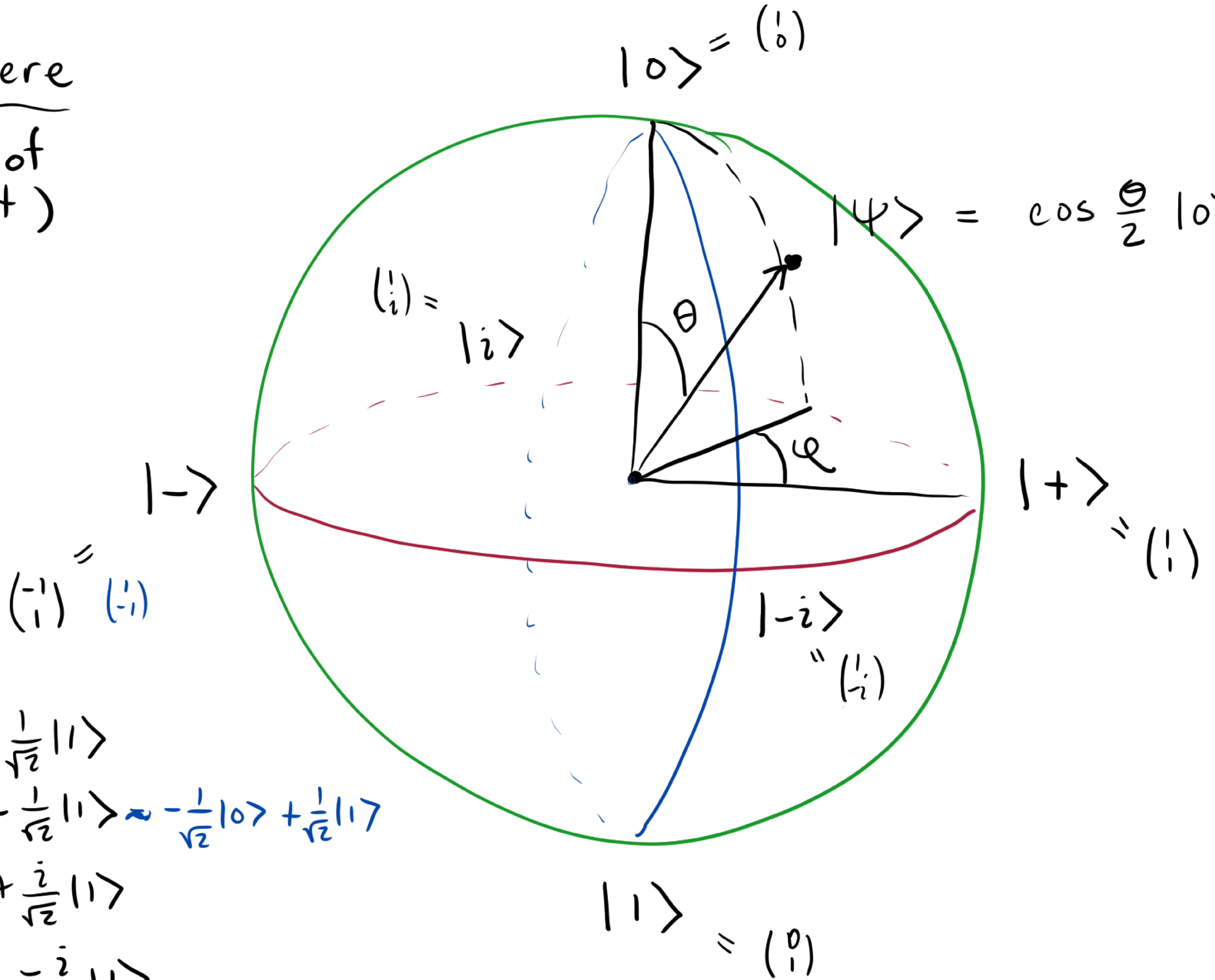
$$|-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle \approx -\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$

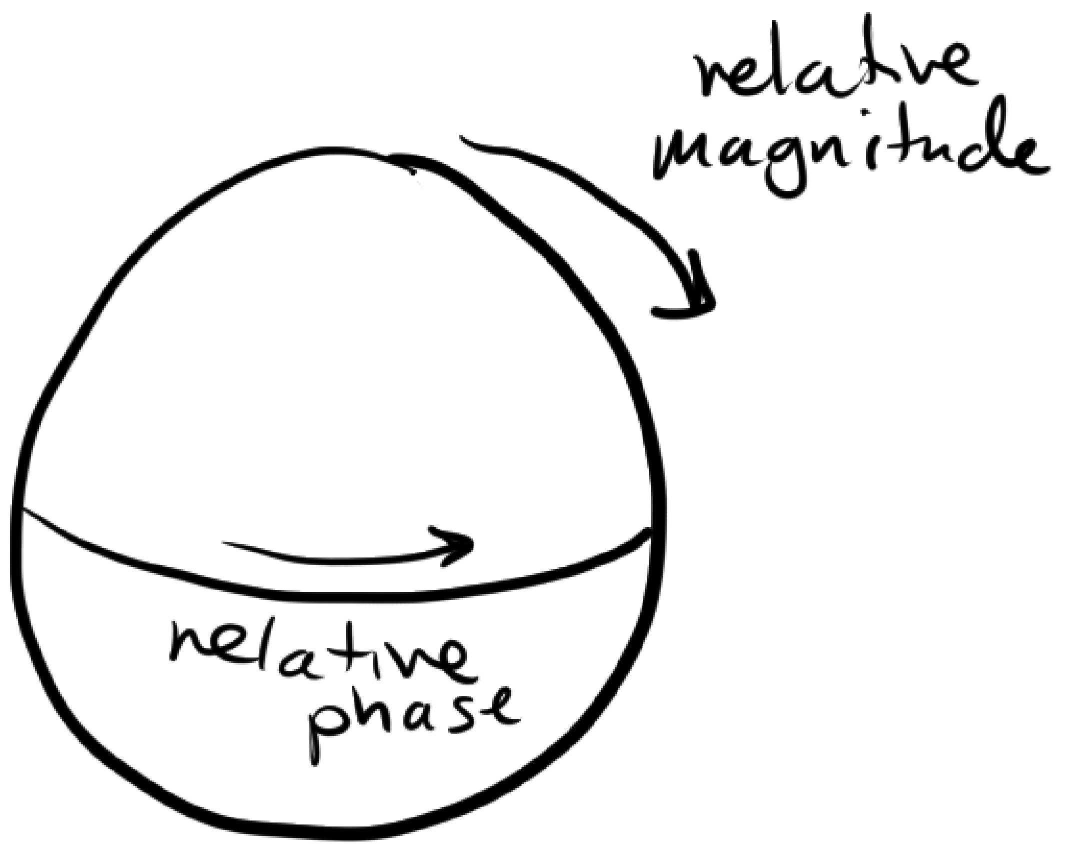
$$|i\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{i}{\sqrt{2}}|1\rangle$$

$$|-i\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{i}{\sqrt{2}}|1\rangle$$

Bloch Sphere

(state space of one qubit)





$$|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

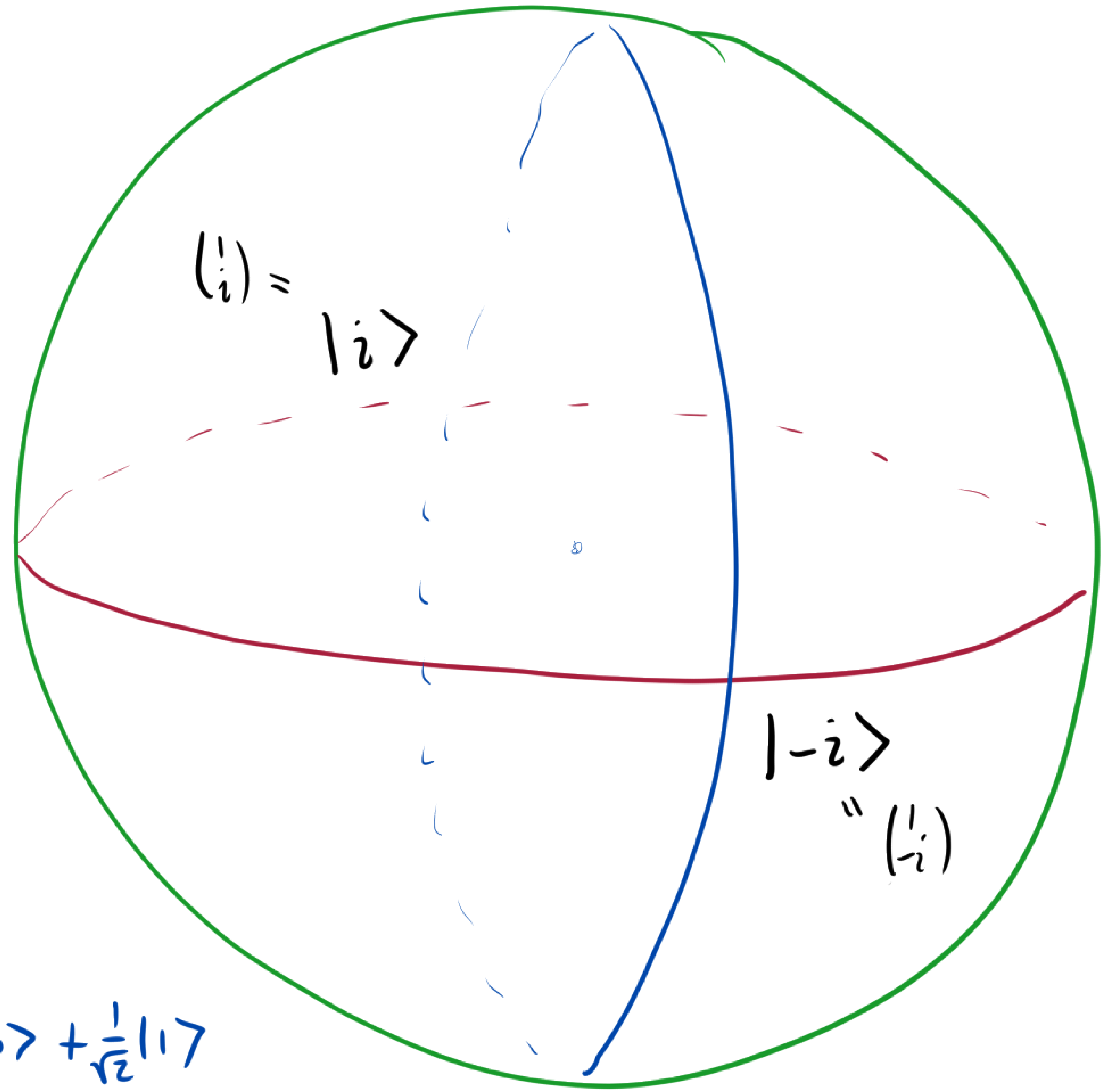
$$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle \approx -\frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$$

$$|i\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle$$

$$|-i\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{i}{\sqrt{2}}|1\rangle$$

Bloch Sphere

(state space of one qubit)



orthogonal bases = antipodal points on sphere

e.g.

$|0\rangle, |1\rangle$

$|+\rangle, |-\rangle$

$|i\rangle, |-i\rangle$

$$|+\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$

$$|-\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{1}{\sqrt{2}}|1\rangle \approx -\tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{1}{\sqrt{2}}|1\rangle$$

$$|i\rangle = \tfrac{1}{\sqrt{2}}|0\rangle + \tfrac{i}{\sqrt{2}}|1\rangle \overset{\times i}{\approx} \tfrac{i}{\sqrt{2}}|0\rangle + \tfrac{-1}{\sqrt{2}}|1\rangle$$

$$|-i\rangle = \tfrac{1}{\sqrt{2}}|0\rangle - \tfrac{i}{\sqrt{2}}|1\rangle$$

Changing Basis

Example. What is $\overset{\begin{pmatrix}1\\0\end{pmatrix}}{|0\rangle}$ in the $\overset{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}}{|+\rangle}$, $\overset{\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}}{|-\rangle}$ basis?

Recall: $|+\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle$

$|-\rangle = \frac{1}{\sqrt{2}}|0\rangle - \frac{1}{\sqrt{2}}|1\rangle$

Solve: $$\begin{pmatrix}1\\0\end{pmatrix} = a\begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix} + b\begin{pmatrix}1/\sqrt{2}\\-1/\sqrt{2}\end{pmatrix}$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 1\\1 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}a\\b\end{pmatrix} = \frac{\sqrt{2}}{(-2)}\begin{pmatrix}-1 & -1\\-1 & 1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1/\sqrt{2}\\1/\sqrt{2}\end{pmatrix}$$

So $$|0\rangle = \frac{1}{\sqrt{2}}|+\rangle + \frac{1}{\sqrt{2}}|-\rangle$$

Measurement (w.r.t. a basis)

Measuring $|\psi\rangle = a|0\rangle + b|1\rangle$ in the basis $|0\rangle, |1\rangle$

returns either $|0\rangle$ or $|1\rangle$:

$$\begin{cases} |0\rangle & \text{with probability } |a|^2 \\ |1\rangle & \text{with \quad '' \quad } |b|^2 \end{cases}$$

and then $|\psi\rangle = |0\rangle$ if it returned $|0\rangle$

$|\psi\rangle = |1\rangle$ if it returned $|1\rangle$

i.e. the state (superposition) collapses to what was measured.

Example. $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|1\rangle = |+\rangle$

In $|0\rangle, |1\rangle$ basis:

result	prob
$\|0\rangle$	$1/2$
$\|1\rangle$	$1/2$

In $|+\rangle, |-\rangle$ basis: $|\psi\rangle = 1 \cdot |+\rangle + 0 \cdot |-\rangle$

result	prob
$\|+\rangle$	1
$\|-\rangle$	0

Example: Polarized Filters

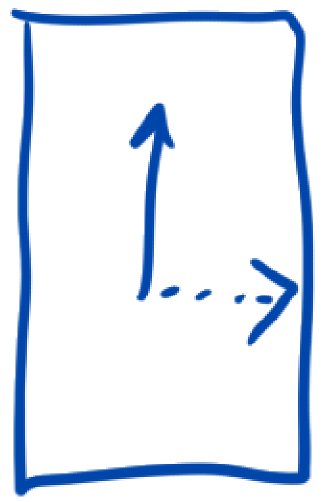
<u>photons = qubits</u>

vertical polarization $|\uparrow\rangle$

horizontal polarization $|\rightarrow\rangle$

} a photon polarization is a superposition.

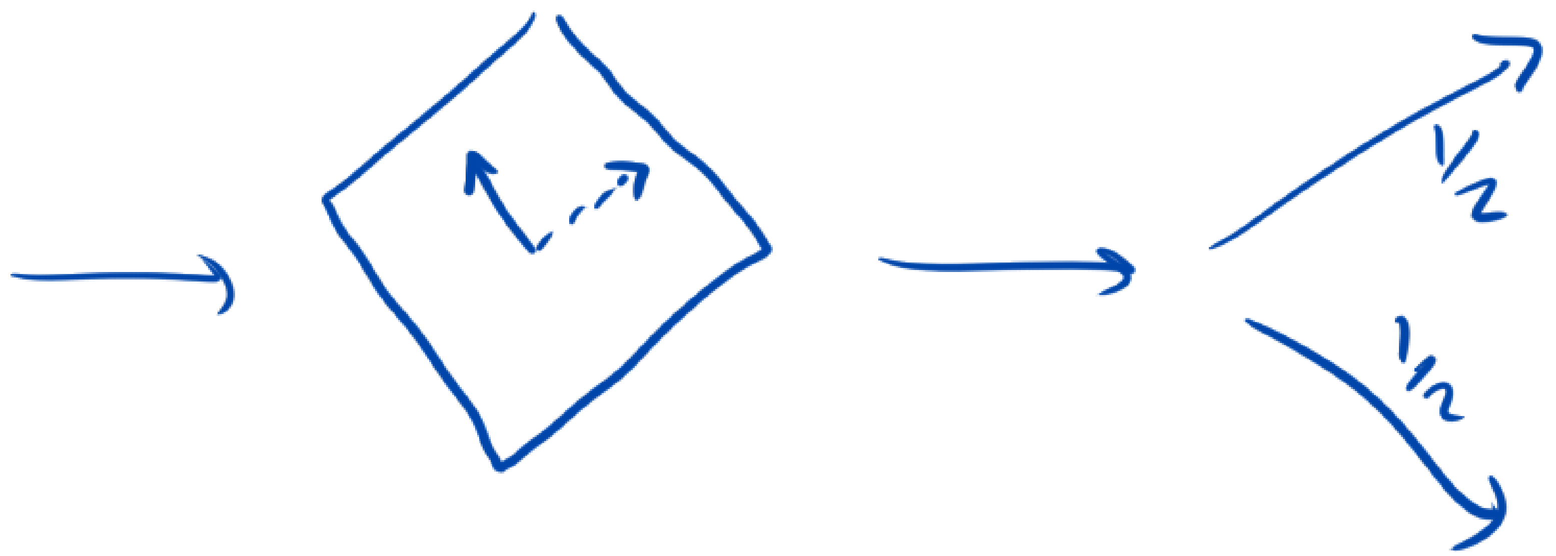
a filter is a measurement device



light passes = observed state $|\uparrow\rangle$

light blocked = " " $|\rightarrow\rangle$

$|\uparrow\rangle$

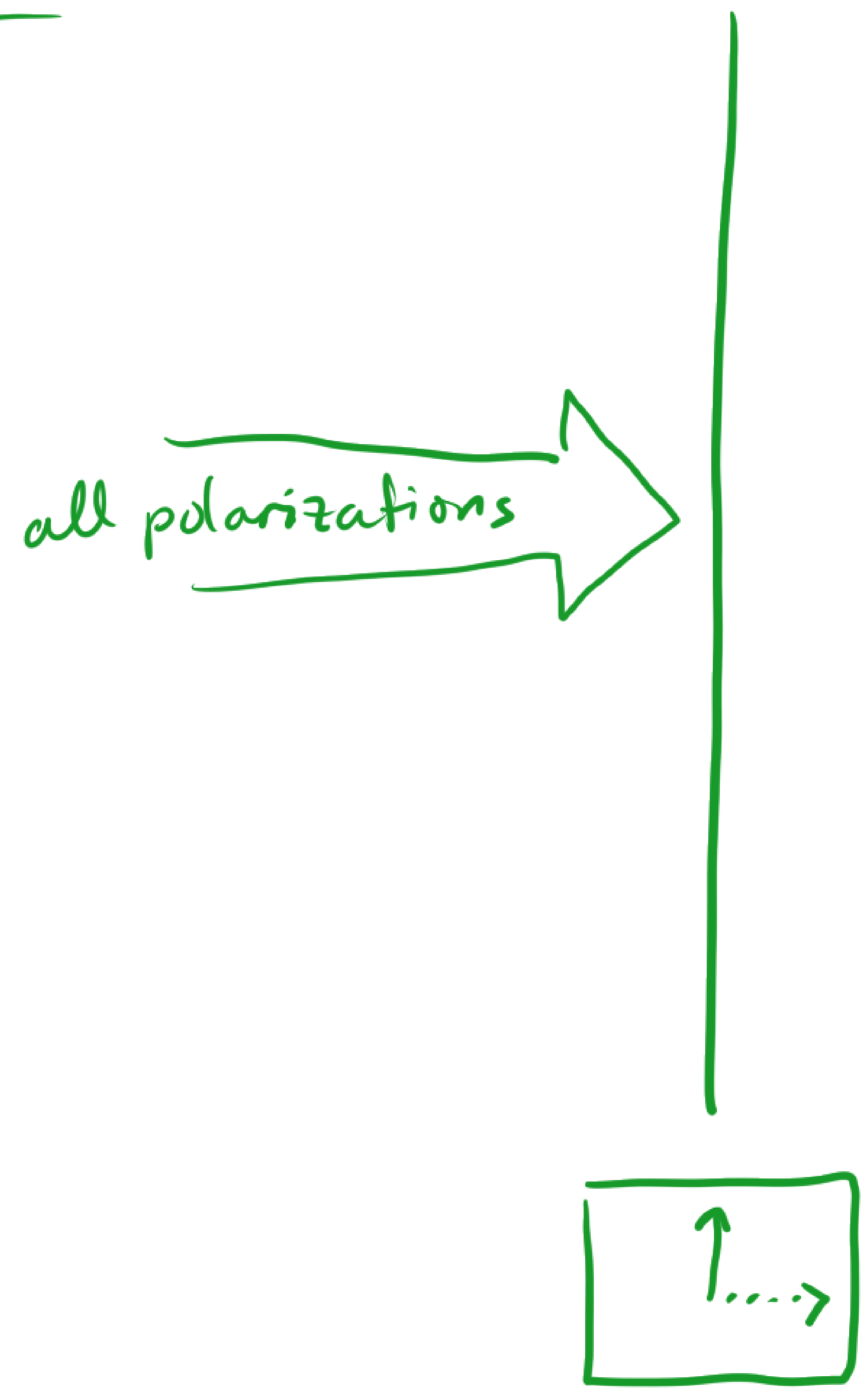


$|\nwarrow\rangle$

blocked $(|\nearrow\rangle)$

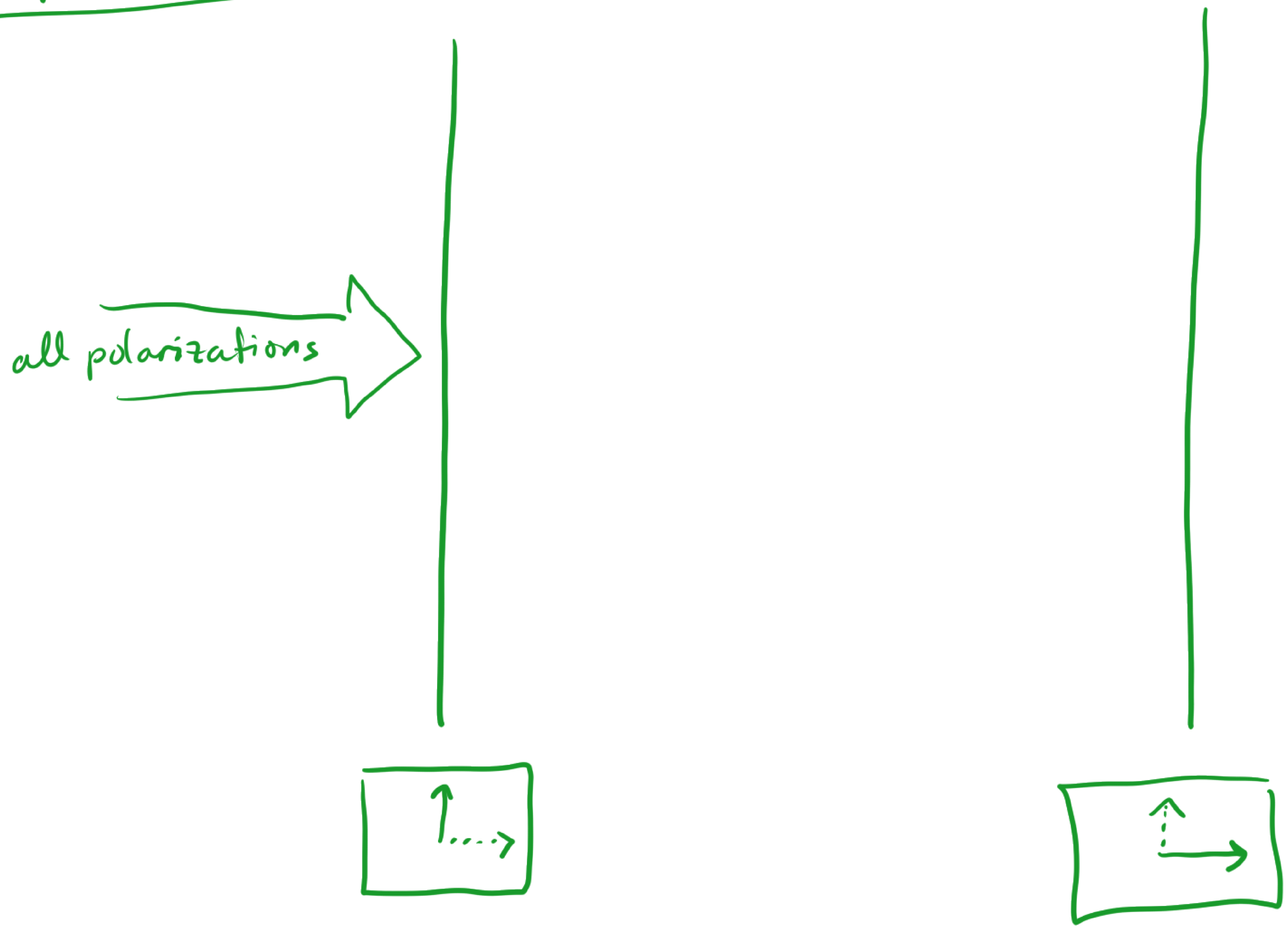
Example: Polarized Filters

Experiment A



Example: Polarized Filters

Experiment B



Example: Polarized Filters

Experiment C