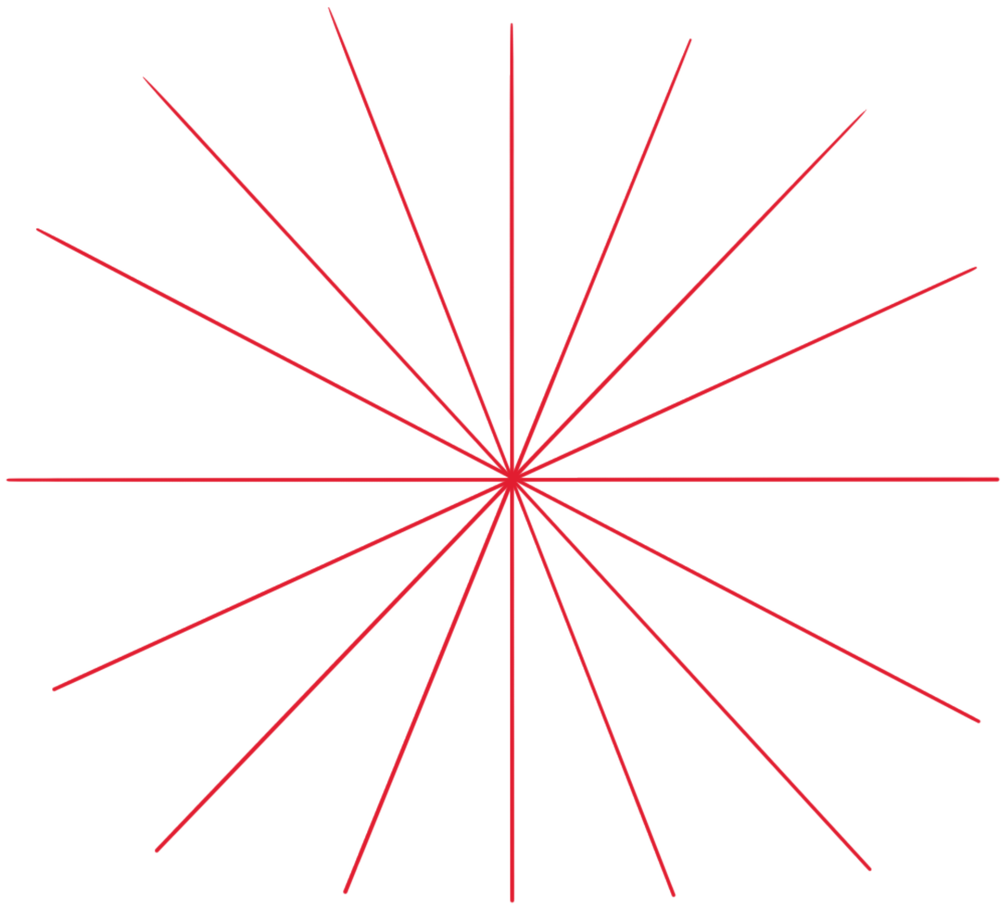


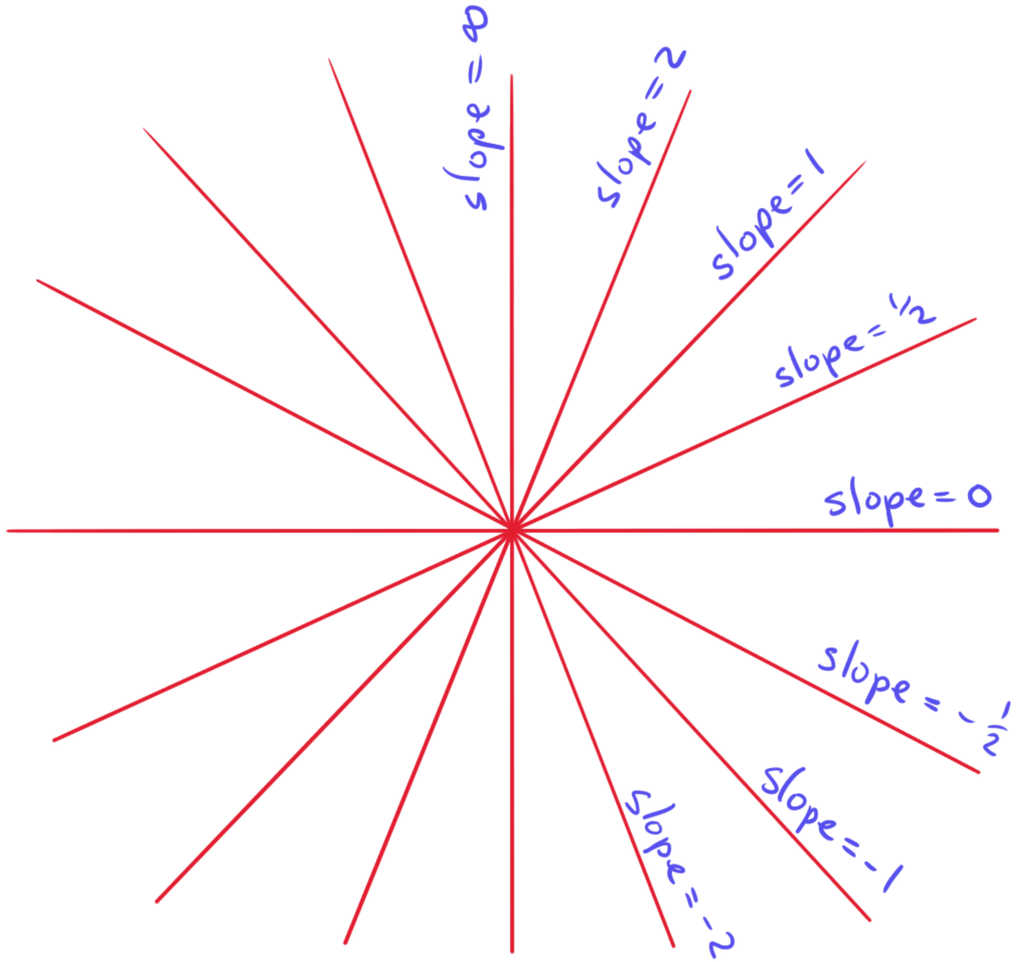
$$\{\text{lines through the origin}\}$$

$$\updownarrow$$

$$\{\text{slopes}\}$$

$$\updownarrow$$

$$\mathbb{R} \cup \{\infty\}$$



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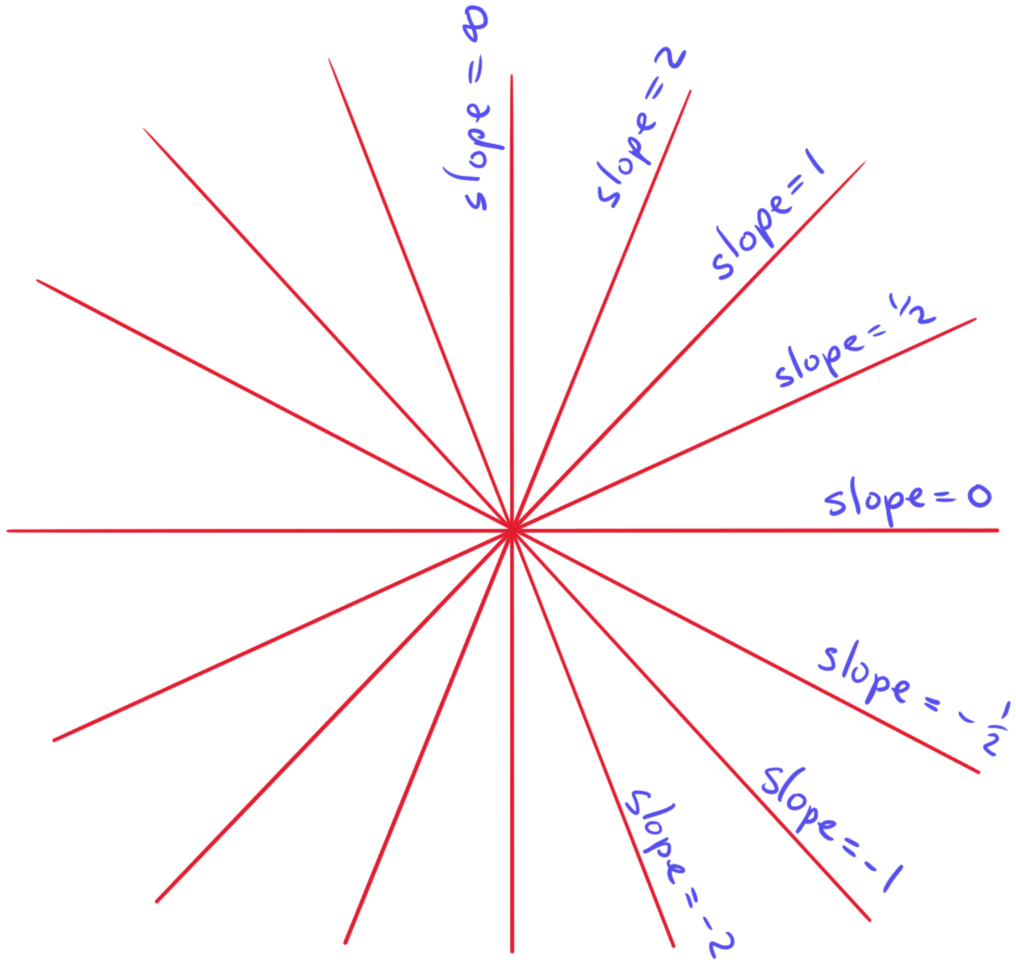

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$\leftrightarrow$

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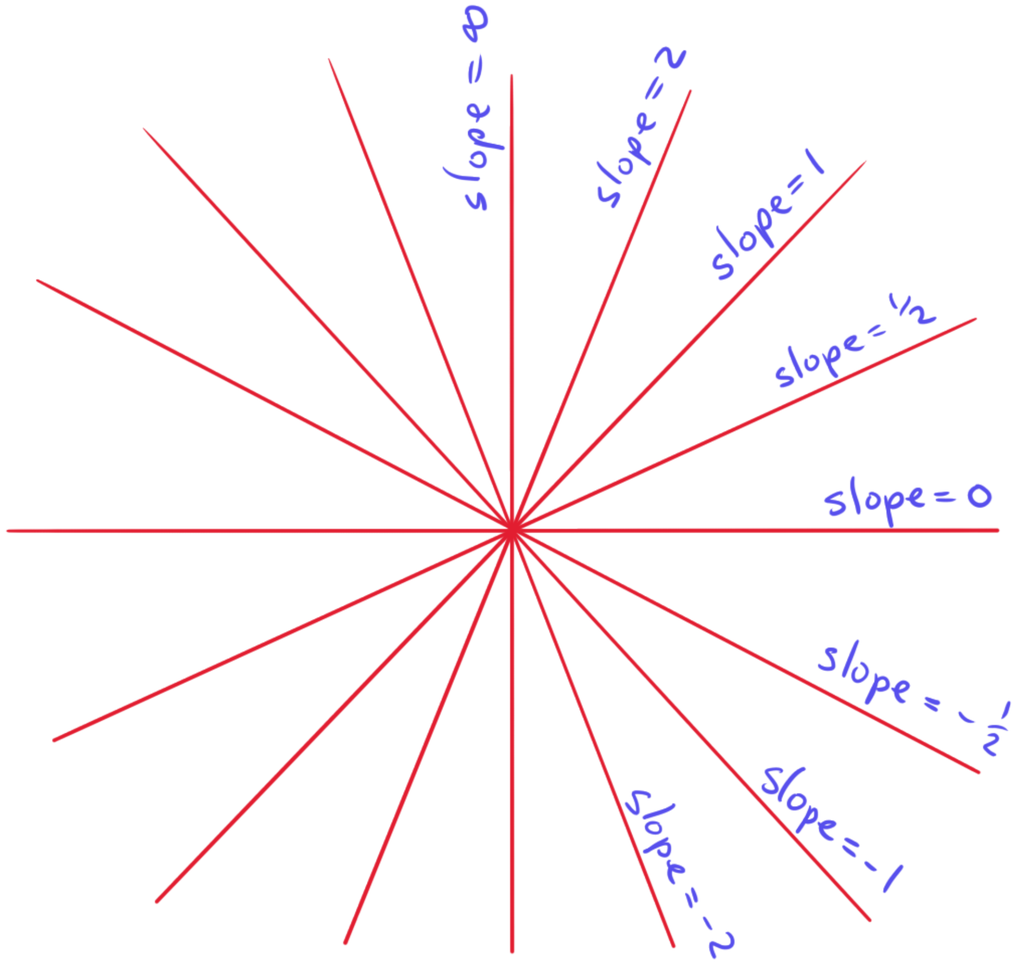
The projective line, denoted $\mathbb{P}^1_{\mathbb{R}}$



$$\mathbb{P}^1_{\mathbb{R}} = \text{vectors in plane modulo length}$$

$$= \{ \vec{v} \in \mathbb{R}^2 \setminus \{\vec{0}\} \} / \sim$$

where

$\vec{v}_1 \sim \vec{v}_2$ iff

$\vec{v}_1 = \lambda \vec{v}_2$

for some $\lambda \in \mathbb{R}^* = \mathbb{R} \setminus \{0\}$

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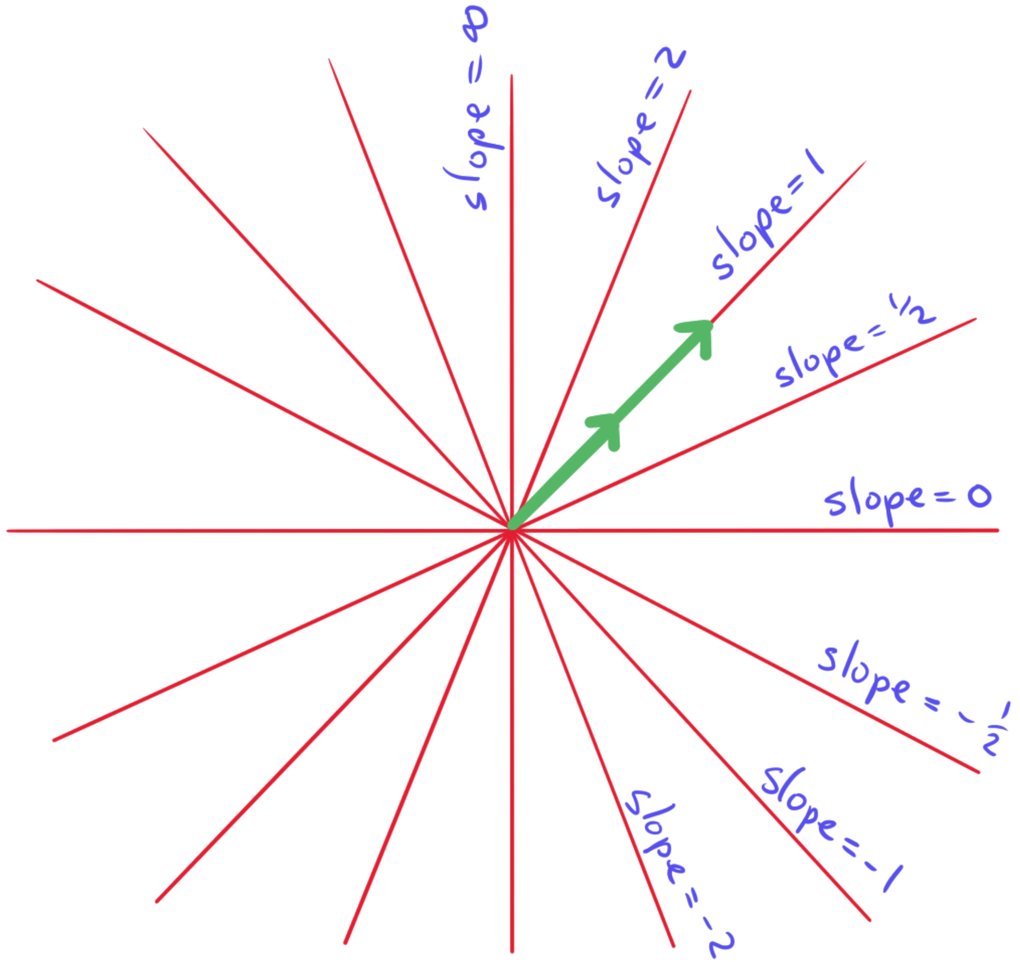

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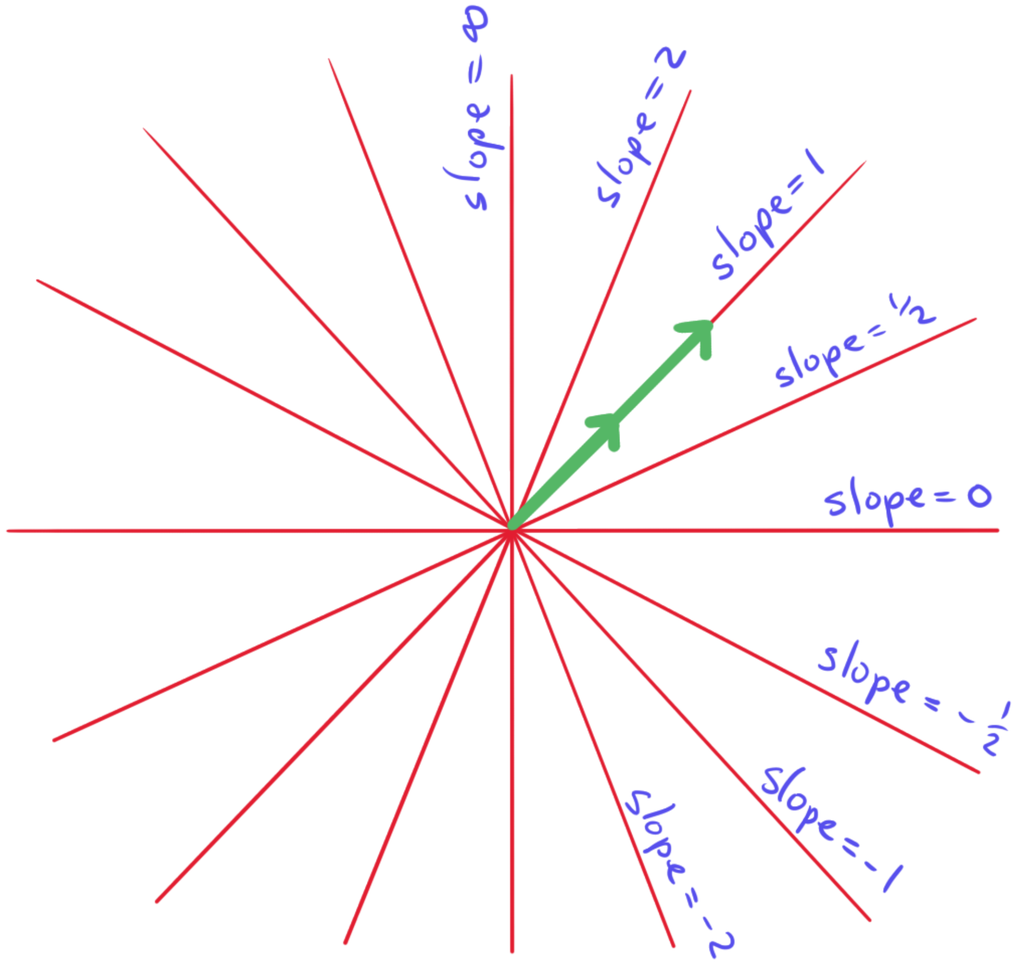
Example: $(1,1) \sim (2,2)$



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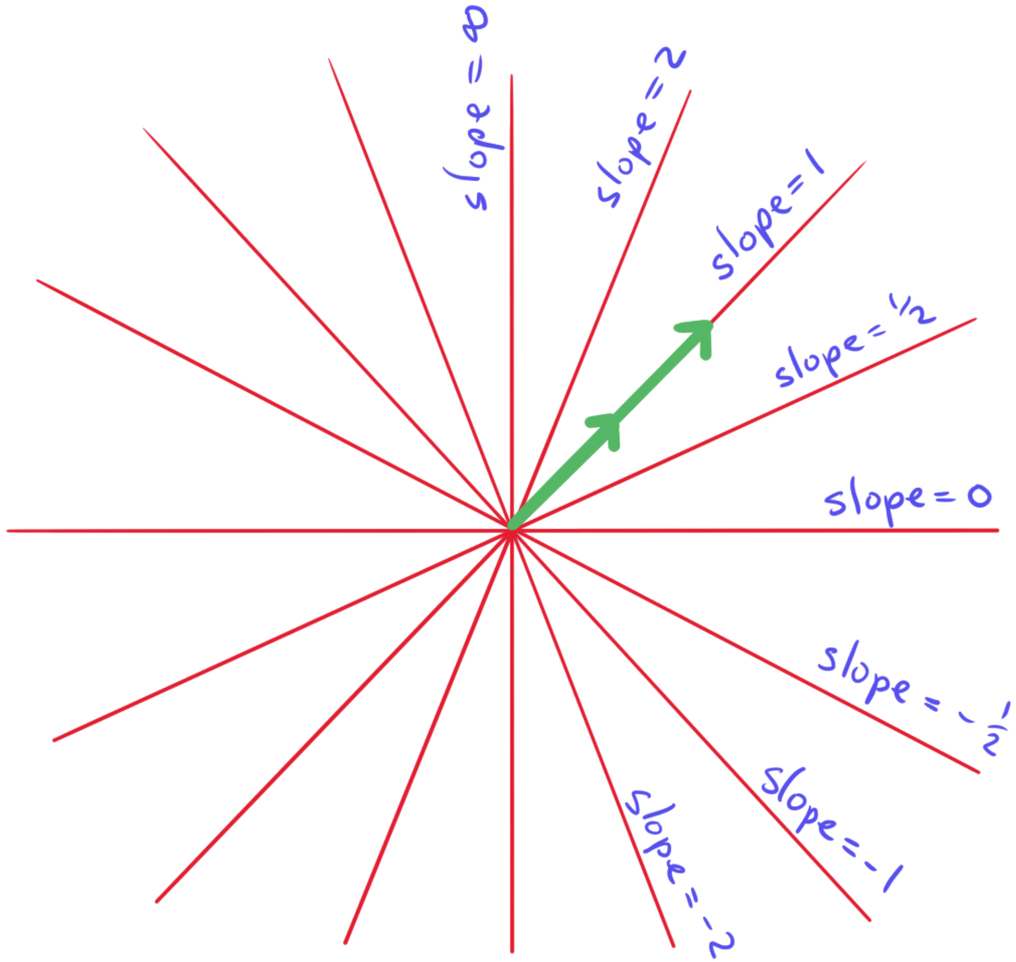

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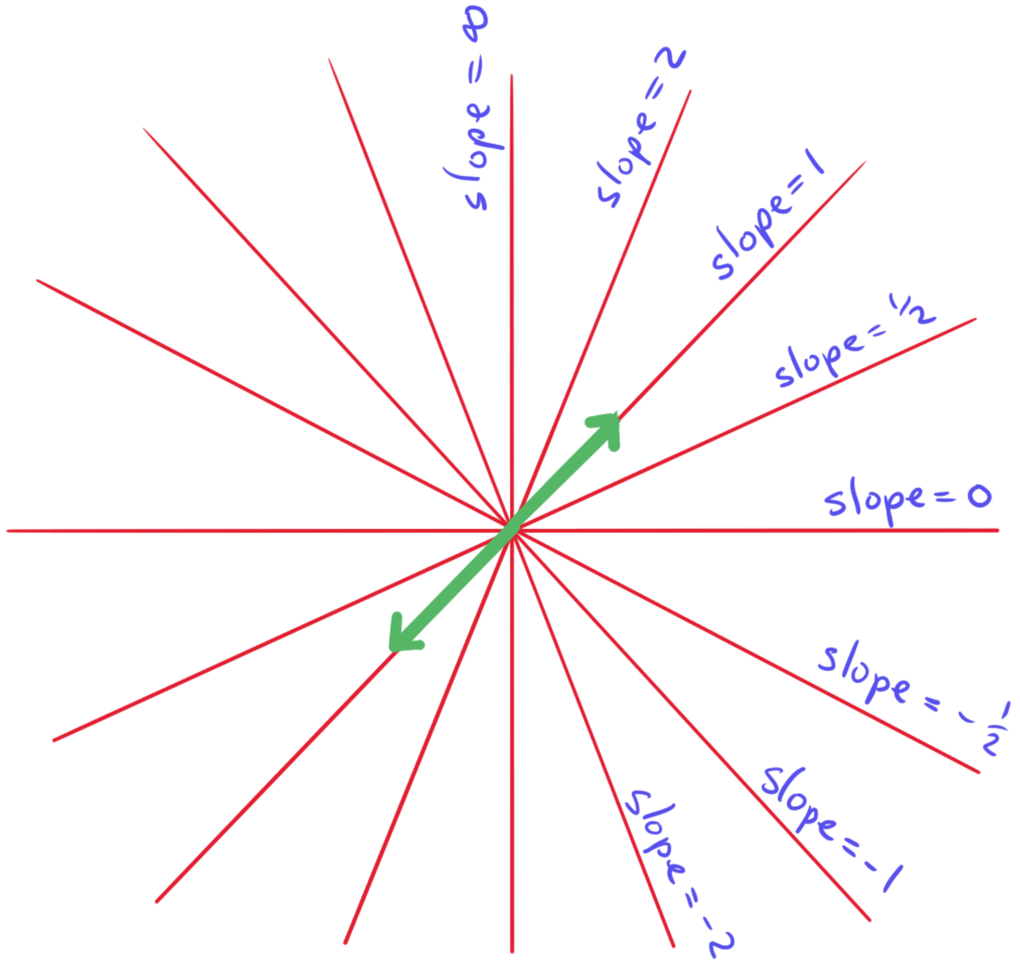

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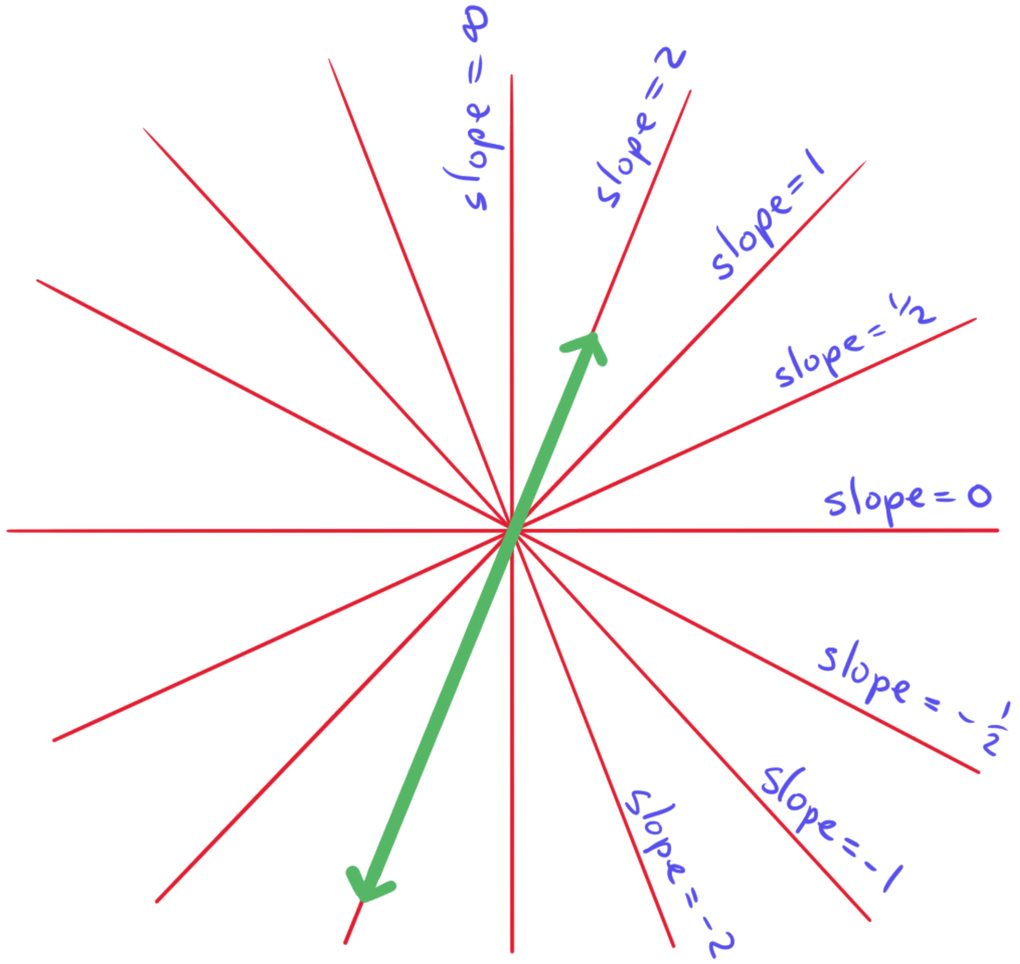
notation: $[1,1] = [2,2] = [-1,-1]$



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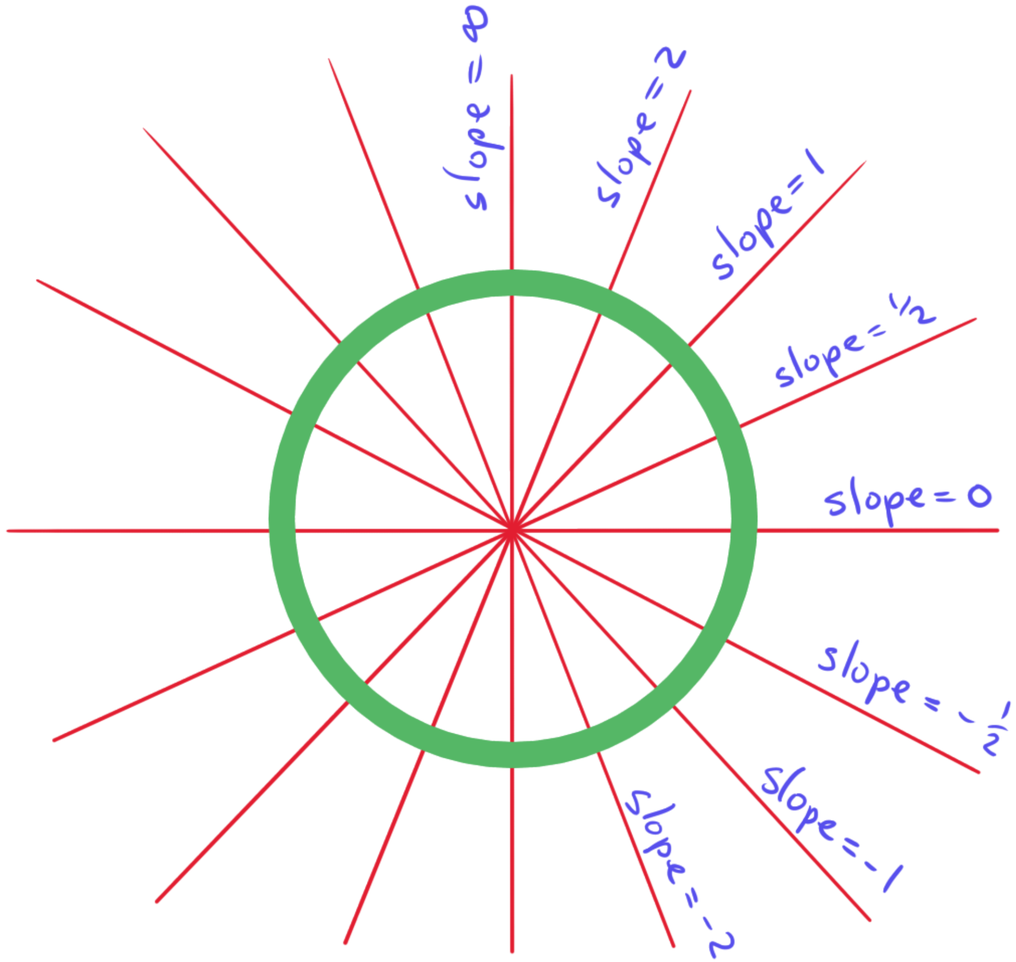
Example: $(\frac{1}{2}, 1) \sim (-1, -2)$

notation: $[\frac{1}{2}, 1] = [-1, -2] = [1, 2]$



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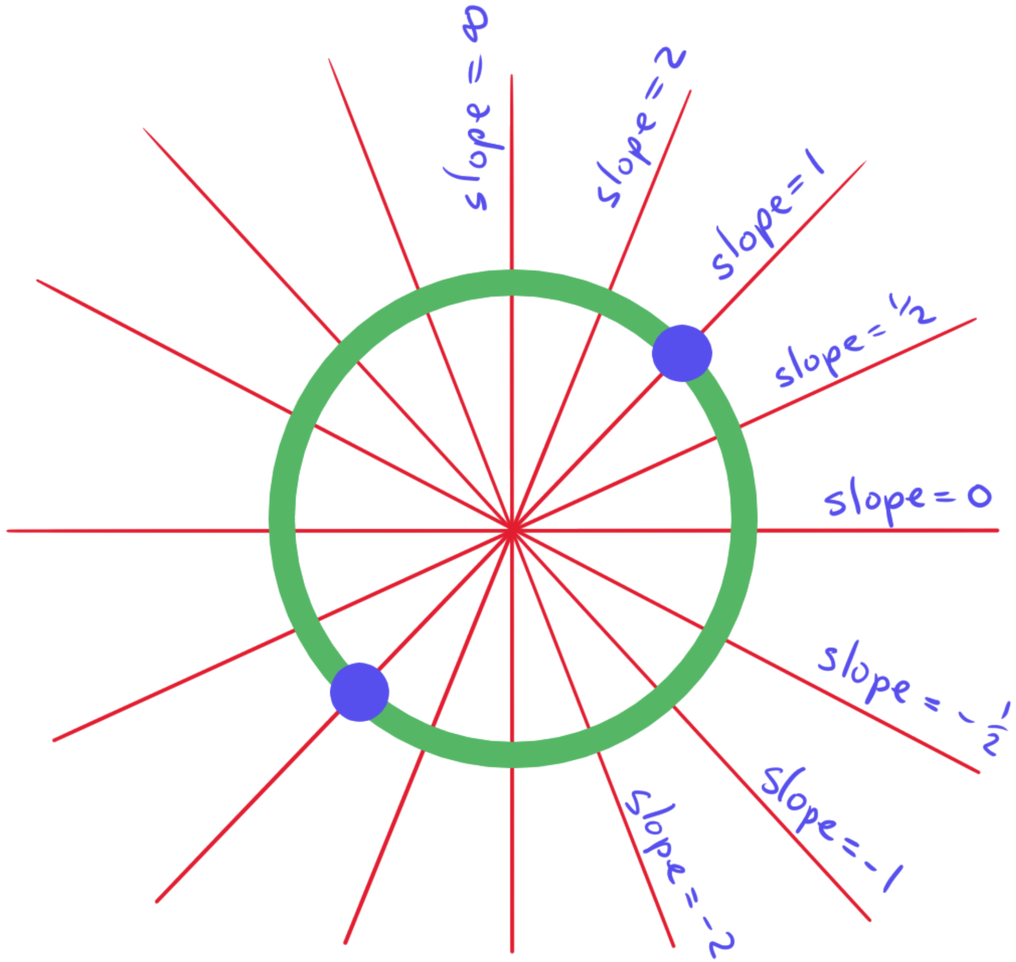

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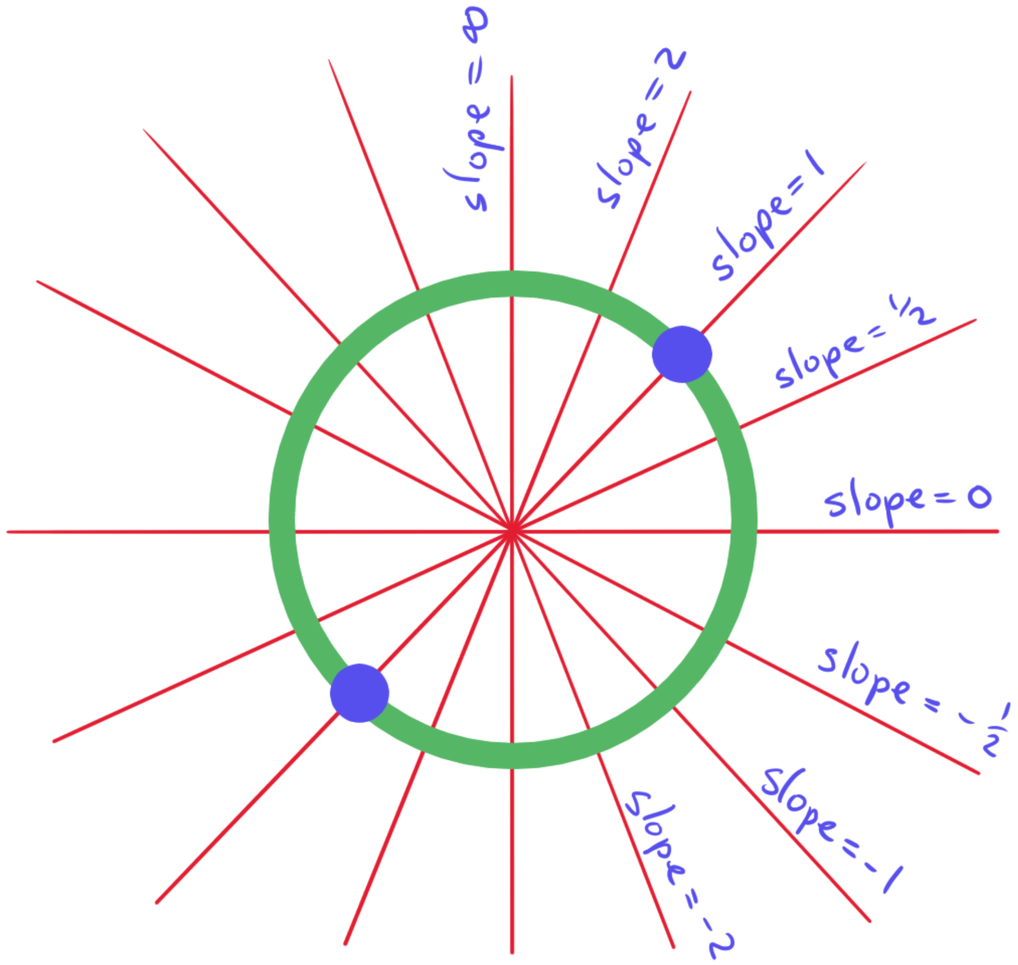

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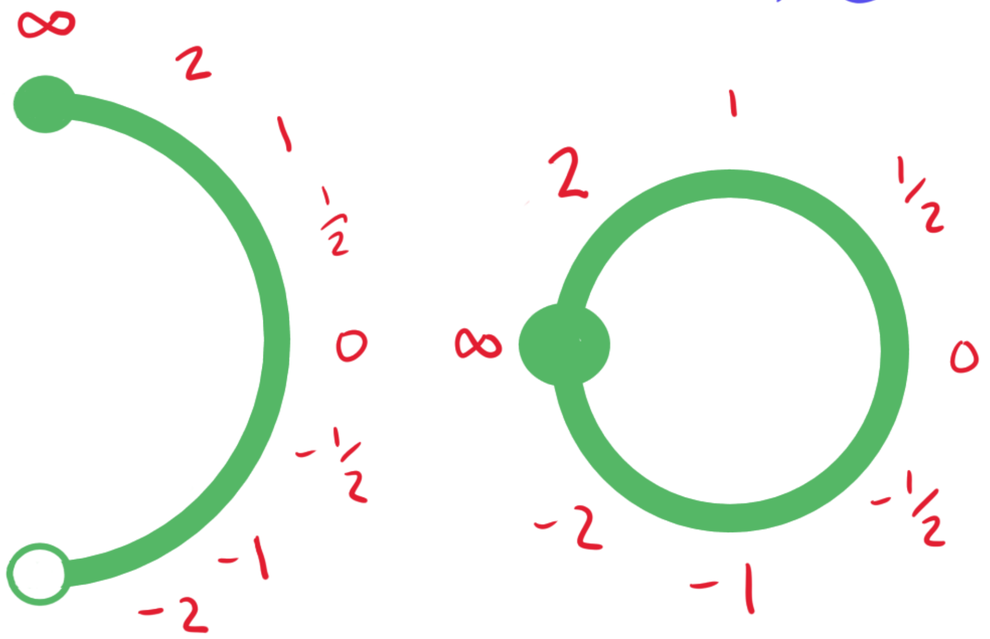
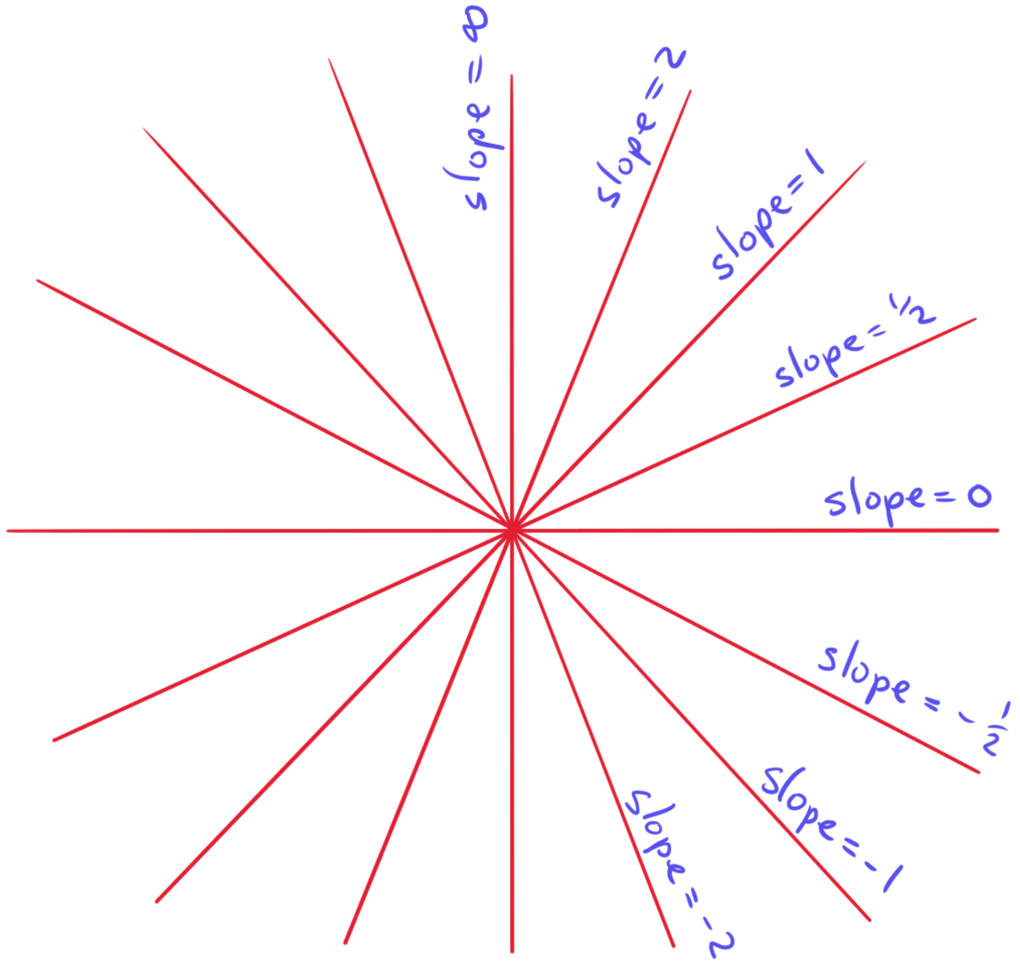


$$\mathbb{R} \cup \{\infty\}$$



$$\mathbb{P}^1_{\mathbb{R}} = \text{vectors in plane modulo length}$$

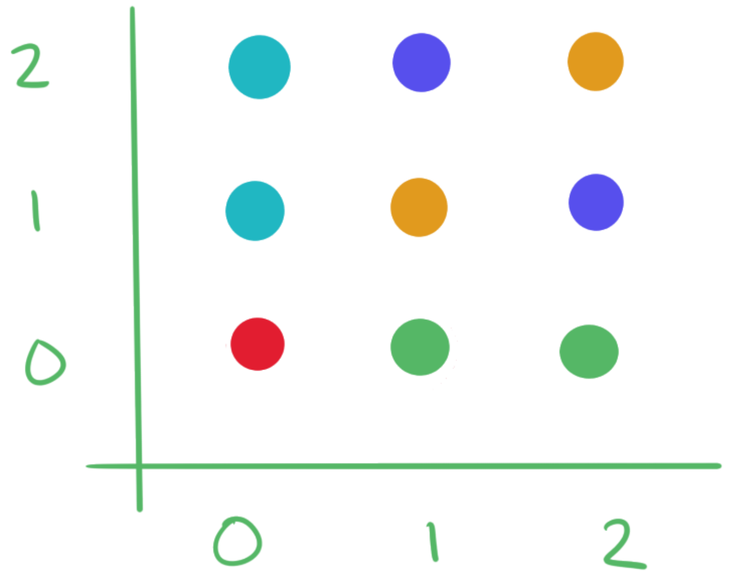
$$= \{ \vec{v} \in \mathbb{R}^2 \setminus \{\vec{0}\} \} / \sim$$

$$\mathbb{P}^1_{\mathbb{F}_p} = \text{vectors in plane modulo length}$$

$$= \left\{ \vec{v} \in \mathbb{F}_p^{\,2} \setminus \{\vec{0}\} \right\} / \sim$$

Example. $\mathbb{P}^1_{\mathbb{F}_3}$

Note: $\mathbb{F}_3^\times = \{1, 2\}$



$[1,0] = [2,0]$
$[1,1] = [2,2]$
$[2,1] = [1,2]$
$[0,1] = [0,2]$

$$\left|\mathbb{P}^1_{\mathbb{F}_3}\right| = 4$$

$$\mathbb{P}^1_{\mathbb{F}_p} = \text{vectors in plane modulo length}$$

$$= \left\{ \vec{v} \in \mathbb{F}_p^{\,2} \setminus \{\vec{0}\} \right\} / \sim$$

$\vec{v}_1 \sim \vec{v}_2$ iff $\vec{v}_1 = \lambda \vec{v}_2$ for some $\lambda \in \mathbb{F}_p^*$

$$\mathbb{P}^1_{\mathbb{R}} = \text{vectors in plane modulo length}$$

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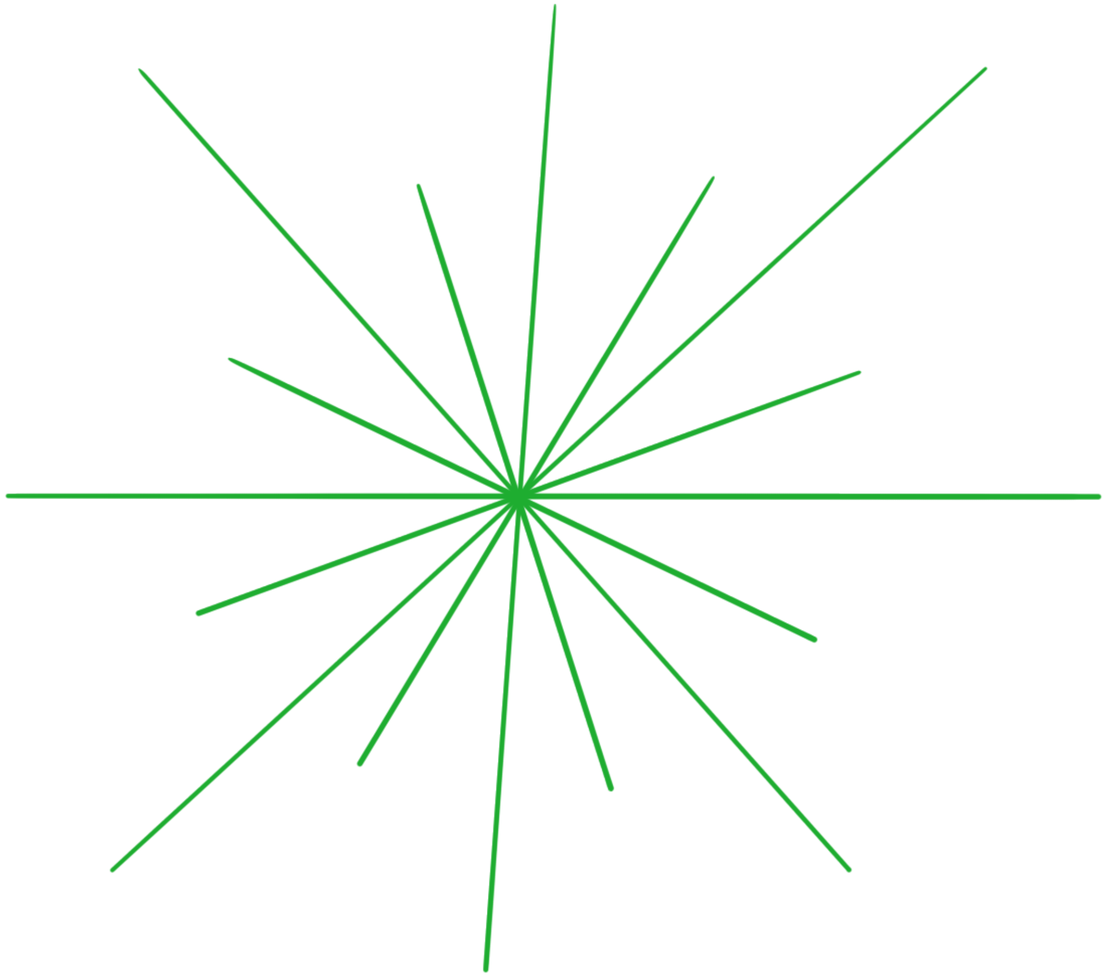

$$\mathbb{P}^2_{\mathbb{R}} = \text{vectors in space modulo length}$$

$$= \{ \vec{v} \in \mathbb{R}^3 \setminus \{\vec{0}\} \} / \sim$$

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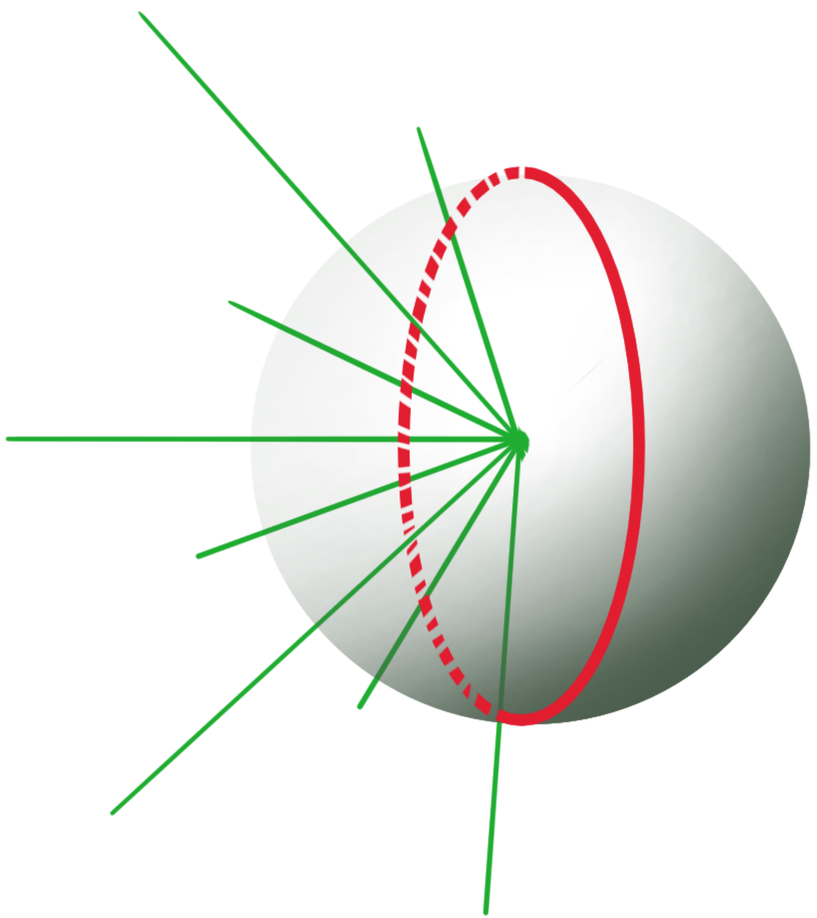

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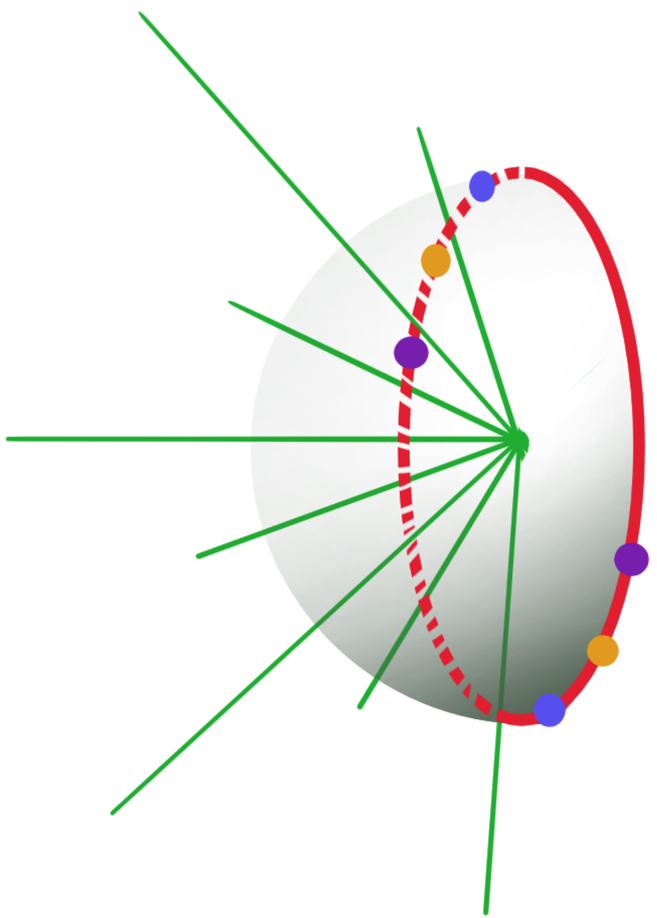

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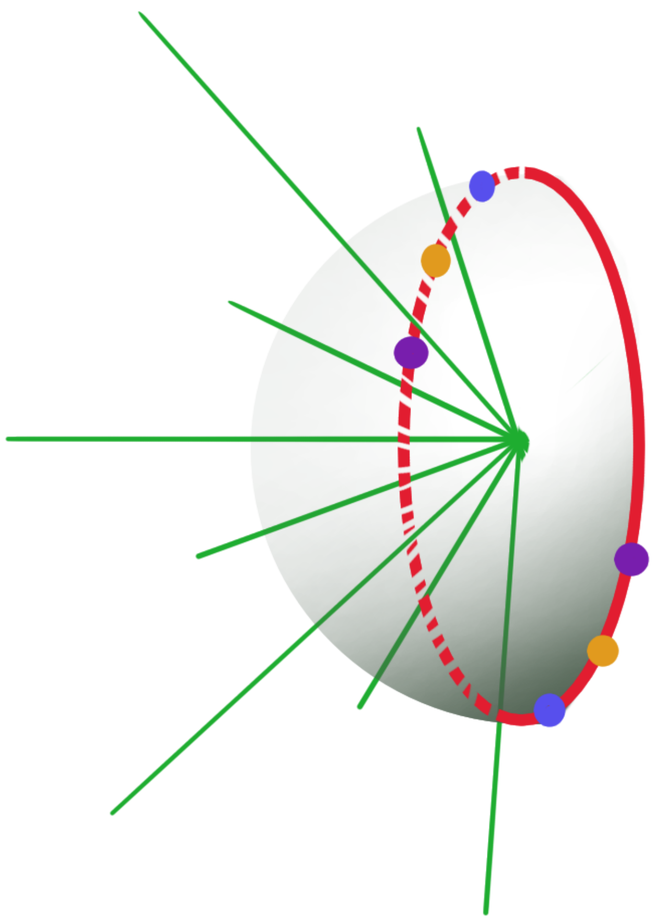

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Example: $(1,0,7) \sim (2,0,14)$ notation: $[1,0,7]$

$$\mathbb{P}^2_{\mathbb{F}_p} = \text{vectors in space modulo length}$$

$$= \{ \vec{v} \in \mathbb{F}_p^3 \setminus \{\vec{0}\} \} / \sim$$

$\vec{v}_1 \sim \vec{v}_2$
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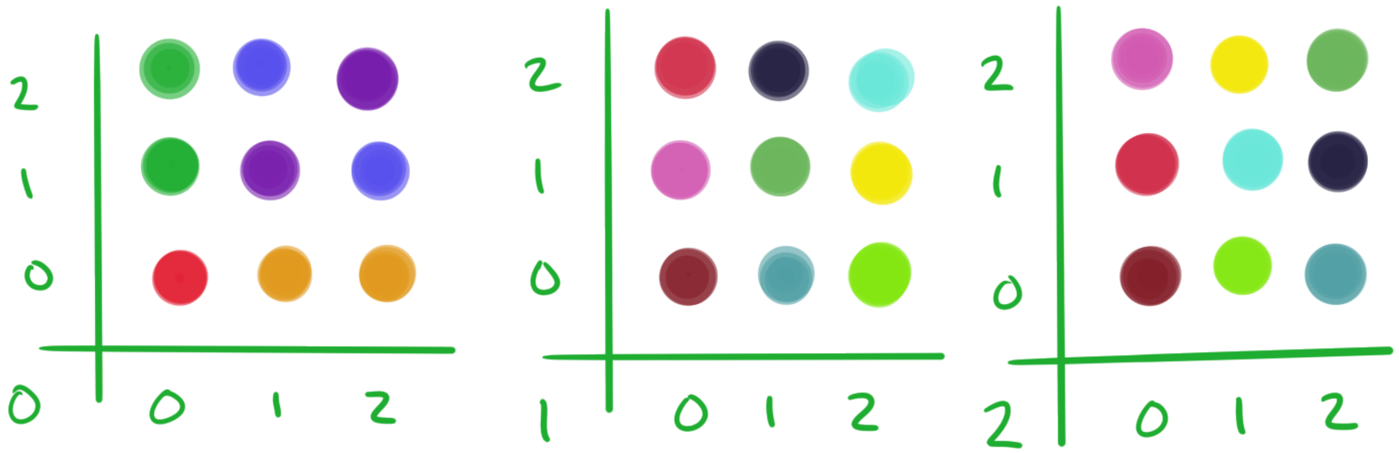
Example. $\mathbb{P}^2_{\mathbb{F}_3}$ = projective plane over $\mathbb{F}_3$



each equivalence class has 2 elements

26 nonzero vectors

$\Rightarrow$ 13 elements in $\mathbb{P}^2_{\mathbb{F}_3}$

Example: $[1,1,2] = [2,2,1]$

$$\mathbb{P}^2_{\mathbb{F}_p} = \text{vectors in space modulo length}$$

$$= \{ \vec{v} \in \mathbb{F}_p^3 \setminus \{\vec{0}\} \} / \sim$$

$\vec{v}_1 \sim \vec{v}_2$ iff $\vec{v}_1 = \lambda \vec{v}_2$ for some $\lambda \in \mathbb{F}_p^*$

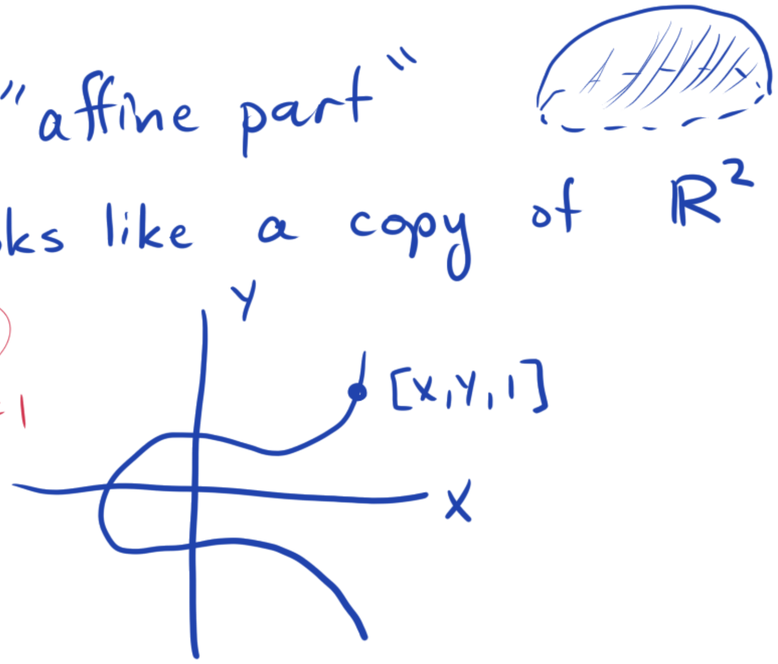
Elliptic Curve
in the
projective
plane

Elliptic Curves in projective space.

$$E: Y^2Z = X^3 + aXZ^2 + bZ^3$$

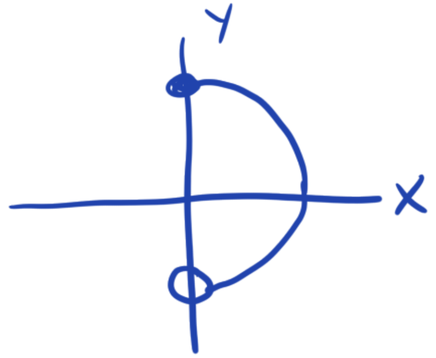
$[X,Y,Z] \in E$ if it satisfies the equation

The "affine part" looks like a copy of $\mathbb{R}^2$



The "line at infinity" is a copy of $\mathbb{P}^1$

$\{[X,Y,0]\}$



The curve has only one pt here:

$[0,1,0]$

$=$ "∞"

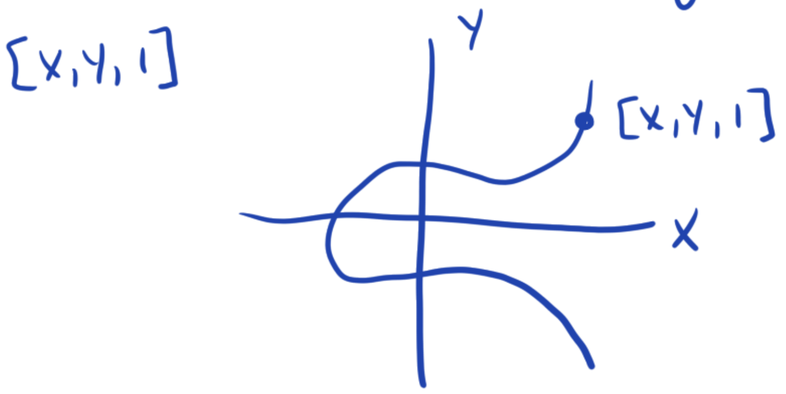
Elliptic Curves in projective space.

$$E: Y^2Z = X^3 + aXZ^2 + bZ^3$$

$[X,Y,Z] \in E$ if it satisfies the equation

The "affine part"  $Z \neq 0$

looks like a copy of $\mathbb{R}^2$



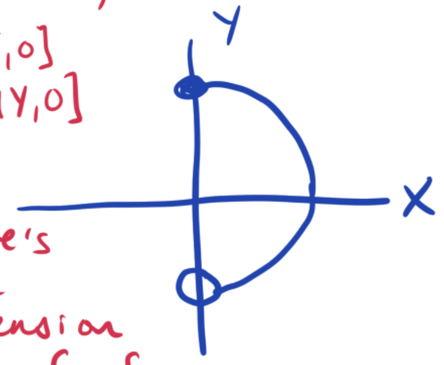
with $z=1$ fixed you can't scale X,Y so there's 2 dimensions (X,Y) of info

The "line at infinity"  $Z=0$

is a copy of $\mathbb{P}^1$

with $z=0$, $\{[X,Y,0]\}$

$[X,Y,0] = [\lambda X, \lambda Y, 0]$ so there's one dimension (slope) of info



The curve has only one pt here:

$[0,1,0]$

$= "\infty"$

Mod 5 $E: y^2 = x^3 + 2x + 4$

$P = (0,2)$ $Q = (0,3)$

$P + Q = \infty$.

Line through 2 pts:

$x = 0$ slope $= \infty$.

slope $= \frac{3-2}{0-0} = \frac{1}{0}$ → crash!

Mod 10 . $P = (0,2)$ $Q = (5,3)$

slope $= \frac{3-2}{5-0} = \frac{1}{5}$

inverse of 5 mod 10

doesn't exist!

<SAGE ERROR>

$\gcd(5,10) = 5$ ← nontrivial factor.