1 Getting to know the Stirling numbers

**Definition 1.** The $(n,k)$ signless Stirling number of the first kind, denoted $c(n,k)$, is equal to the number of permutations of $[n]$ having exactly $k$ cycles.

1. Compute the following:
   \[
   c(3,0) = \\
   c(3,1) = \\
   c(3,2) = \\
   c(3,3) = \\
   c(3,4) = \\
   c(4,2) = \\
   
   
2. Describe the general patterns:
   \[
   c(n,0) = \\
   c(n,1) = \\
   c(n,n) = \\
   
   If $k > n$, then $c(n,k) =$

2 Recurrence

Give a combinatorial proof of the following recurrence:

\[
c(n,k) = c(n-1,k-1) + (n-1)c(n-1,k).
\]
3 Generating Function

Let

\[ G_n(x) := \sum_{k=0}^{\infty} c(n,k)x^k. \]

1. Why is \( G_n(x) \) a polynomial?

2. Compute the following polynomials, and also give them in factored form

\[ G_1(x) = \]
\[ G_2(x) = \]
\[ G_3(x) = \]

3. Write a conjecture, based on the data above: \textit{The generating function } \( G_n(x) \) \textit{is equal to the polynomial } \( P_n(x) \) \textit{defined by } \( P_n(x) = \)

4. Based on your definition of \( P_n(x) \) above, give a recurrence relation for \( P_n(x) \) in terms of \( P_{n-1}(x) \).

5. Using the recurrence relation of the last section, conjecture a recurrence for \( G_n(x) \) in terms of \( G_{n-1}(x) \).
6. Verify this recurrence by direct computation using the definition of $G_n(x)$.

7. Use the work you’ve done to show that $G_n(x) = P_n(x)$. 
4 \textbf{Something else}

Permutations are functions, so they can be composed.

1. Compose a few permutations to get the feel for it.

2. Does composition commute?

3. Let $\sigma^k$ denote composition of a permutation $\sigma$ with itself $k$ times. Prove that for any permutation of $[n]$, there is some integer $k > 0$, so that $\sigma^k$ is the identity.