Graph Theory

November 15, 2015

1 Warmup: Cycle graphs

Definition 1. *The* cycle graph C_n *is the graph given by the following data:*

$$\begin{aligned} V_G &= \{v_1, v_2, \ldots, v_n\} \\ E_G &= \{e_1, e_2, \ldots, e_n\} \\ \epsilon(e_i) &= \{v_i, v_{i+1}\}, \end{aligned}$$

where the indices in the last line are interpreted modulo n.

1. Draw C_n for $n = 0, 1, 2, 3, 4, 5$.

2. Is C_n simple?
3. Is C_n bipartite?
4. Does C_n have an Eulerian circuit? How many?
5. Does C_n have a Hamiltonian cycle? How many?
6. Is C_n connected? Prove it (from the definition).

7. How many walks of length 3 are there in C_n?
8. How many paths of length 3 are there?
9. How many trails of length 3 are there?
10. Compute the adjacency matrix A of C_n.

11. Without doing any matrix multiplications, compute A^{n-1}.

12. Suppose the edge e_i is given the weight i. What spanning tree does one obtain from the greedy algorithm?

13. How many spanning trees does C_n have (from the definition).

14. Prove the last item using the Matrix Tree Theorem we saw in class. (Hint: figure out a recursion for the determinants.)

2 Bipartite graphs

Definition 2. *The* bipartite graph $K_{n,m}$ *is the graph given by the following data:*

$$\begin{aligned} V_G &= \{v_1, v_2, \ldots, v_n\} \cup \{w_1, w_2, \ldots, w_m\} \\ E_G &= \{e_{i,j} : 1 \leq i \leq n, 1 \leq j \leq m\} \\ \epsilon(e_{i,j}) &= \{v_i, w_j\} \text{ for all } 1 \leq i \leq n, 1 \leq j \leq m. \end{aligned}$$

1. Draw $K_{1,1}$, $K_{2,3}$, $K_{4,2}$ and $K_{0,1}$.

2. Does $K_{n,m}$ have an Eulerian circuit?

3. Does $K_{n,m}$ have a Hamiltonian cycle?

4. Compute the adjacency matrix of $K_{n,m}$.

5. Compute the number of paths of length 1.

6. Compute the number of paths of length 2.

7. Prove that $K_{n,m}$ is connected. (Any method.)

8. Count the number of spanning trees of $K_{n,m}$.