Combinatorial Game Theory

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1 Colouring Terminology

Definition 1. *A* colouring *of a graph G by a set S is a function $c : V_G \to S$ such that if $\epsilon^{-1}(u, v) \neq \emptyset$ for $u \neq v \in V_G$, then $c(u) \neq c(v)$.*

In other words, think of S as a set of colours. A colouring is a way to colour each vertex so adjacent vertices are not the same colour.

Definition 2. *Let G be a graph. If there exists a colouring of G by a set of size k, we say that G is k-*colourable*. The smallest n such that G is n-colourable is called the* chromatic number*, denoted $\chi(G)$.*

In other words, the chromatic number is the least number of colours needed to colour the graph.

Definition 3. *A complete subgraph (of an undirected graph) of size n is called a* clique *of size n.*

2 Game Theory Terminology

Consider a deterministic game (no chance), in which each of two players has some finite number of choices on his turn. At any moment the game is in a particular *position* and the player to move can see all relevant information.

A position in our example game consists of a graph with vertices coloured all empty, blue or red. To start, the vertices are all empty. At the first move, one empty vertex is coloured blue. At the second move, one empty vertex is coloured red, etc., alternating colours. The rule is that you may not colour a vertex in such a way that it results in the same colour on two adjacent vertices. If a player is unable to move, he loses and the other player wins.

Definition 4. *A position is a* winning position *if the first player can always win, no matter what his opponent does, provided the first player makes the right moves. Otherwise it is a* losing position*.*

We can draw a *game tree* showing all the possible ways the game may play out. For example, I've handed out a game tree for a linear graph of four vertices.

3 For Today

Play some colouring games on various graphs. Your goal is to prove as much as you can about which positions are winning positions. You may find induction helpful here.