Worksheet on Generating Functions

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This worksheet is adapted from notes/exercises by Nat Thiem.

1 Derivatives of Generating Functions

1. If the sequence $a_0, a_1, a_2, \ldots$ has ordinary generating function $A(x)$, then what sequence has ordinary generating function $A'(x)$?

2. Compute the derivative of $\frac{1}{1-x}$ with respect to x (this is a pure calculus question).

3. Now expand the result as an infinite series in powers of x.

4. Combine the last three parts to prove that

$$\binom{-2}{n}(-1)^n = (n+1).$$

   (note: this can be proven more directly; the point is to illustrate the use of generating functions)

5. If the sequence $e_0, e_1, e_2, \ldots$ has exponential generating function $E(x)$, then what sequence has exponential generating function $E'(x)$?

2 Products of Ordinary Generating Functions

1. Suppose $A(x)$ is the ordinary generating function for $a_0, a_1, a_2, \ldots$ and $B(x)$ is the ordinary generating function for $b_0, b_1, b_2, \ldots$. Write down the sequence having $A(x)B(x)$ as ordinary generating function.

2. Given an ordinary generating function $A(x)$ for a sequence $a_0, a_1, a_2, \ldots$, what sequence has ordinary generating function $\frac{1}{1-x}A(x)$?

3 Products of Exponential Generating Functions

1. Suppose $E(x)$ is the exponential generating function for $e_0, e_1, e_2, \ldots$ and $F(x)$ is the exponential generating function for $f_0, f_1, f_2, \ldots$. Write down the sequence having $E(x)F(x)$ as exponential generating function.

2. Suppose $E(x)$ is the exponential generating function for a sequence $e_0, e_1, e_2, \ldots$. What sequence has generating function $e^x E(x)$?

3. Use the last problem to figure out what sequence has $\frac{e^x}{1-x}$ as its **exponential** generating function.

4. Show that $2^n = \sum_{m=0}^{n} \binom{n}{m}$. Hint: Compute e^{2x} as a series directly and as a product of known generating functions, and compare.

4 An example we know

1. What sequence has ordinary generating function $\frac{1}{(1-x)^k}$? It is a sequence we have studied in this class.

2. Prove the last in another way. Hint: you could use binomial theorem, or you could use the techniques we used to describe the generating function for p_n.