Overview of Generating Functions

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A function associated to a sequence $a_0, a_1, a_2, \ldots$

Two Types

1. Ordinary generating functions $\sum_{n=0}^{\infty} a_n x^n$.
2. Exponential generating functions $\sum_{n=0}^{\infty} a_n \frac{x^n}{n!}$.

Examples

Fill in the table with all the sequences you've learned about generating functions for (some we may not have seen information of all columns), e.g. Catalan numbers, Fibonacci, etc. As you do each, review the way the closed form generating function was derived.

sequence	ordinary	closed form ordinary	exponential	closed form exponential
$1, 1, 1, 1, 1, \ldots$	$\sum_{n=0}^{\infty} x^n$	$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} \frac{x^n}{n!}$	e^x
$1, 2, 3, 4, 5, \ldots$				
$\binom{c}{n}$				

Transformations

Performing some transformation on a generating function changes the sequence it is associated to.

Suppose $A(x)$ (ordinary) and $E(x)$ (exponential) are associated to $a_0, a_1, a_2, \ldots$. Then give the sequence associated to each of these generating functions:

function	sequence	explanation
$A(x)x$	$a_1, a_2, \ldots$	shift once to the left (a_0 deleted)
$A'(x)$		
$\frac{A(x)}{1-x}$		
$E(x)x$		
$E'(x)$		
$E(x)e^x$		

Products of generating functions

You've now seen closed formulas for the products of ordinary and exponential generating functions. Record them here:

Now, as in the proof of Binomial Theorem or the proof of the generating functions for odd and distinct partitions, taking products of generating functions *means something* in terms of what you are counting, specifically:

Theorem 1 (Stated schematically)**.** *If $A(x)$ is the generating function for a sequence a_i that counts how many ways you can do P (depending on n), and $B(x)$ is the generating function for a sequence b_i that counts how many ways you can do Q (depending on m), then $A(x)B(x)$ is the generating function for the number of ways you can do P and Q depending on $n+m$.*

Example: Suppose you want to pick some of 3 distinct donuts, then some of 5 distinct flowers. How many ways can you end up with k objects?

Picking donuts: $(1+x)^3$, $\binom{3}{0}, \binom{3}{1}, \binom{3}{2}, \binom{3}{3}$

Picking flowers: $(1+x)^5$, $\binom{5}{0}, \ldots, \binom{5}{5}$

Picking both: $(1+x)^3(1+x)^5$

Strategies for finding a closed form generating function

Study these examples carefully.

1. Create the sequence by 'transformations' on known sequences, e.g. $1, 2, 3, 4, \ldots$ by a derivative.
2. Derive a linear equation the generating function satisfies, e.g. linear recurrence sequences like Fibonaccis.
3. Derive a higher degree equation the generating function satisfies, e.g. Catalan numbers.
4. Derive a differential equation the generating function satisfies, e.g. Bell numbers (exponential generating function).
5. Describe the multiplication of factors, e.g. binomial theorem proof, or generating function for partitions.

Things generating functions can be good for

1. Getting a closed formula for a sequence (e.g. Fibonacci, Catalan)
2. Proving relations (e.g. odd and distinct partitions, various binomial identities as a consequence of Binomial Theorem)
3. You can use them to estimate growth of a sequence by behaviour of the function (we haven't studied this).
4. Many more we haven't studied!