Worksheet on Inclusion-Exclusion

October 11, 2015

This is a long worksheet and it will probably span two days. Might I suggest that you refrain from working on it between the classes so you can enjoy the discovery collaboratively.

1 A Combinatorial Proof

Our goal is to prove the following formula:

$$\sum_{i=0}^{\lfloor \frac{k-1}{2} \rfloor} \binom{k}{2i+1} = \sum_{i=0}^{\lfloor \frac{k}{2} \rfloor} \binom{k}{2i}$$

The symbol $\lfloor x \rfloor$ means the largest integer $\leq x$. In other words, for example, $\lfloor 5 \rfloor = 5$ but $\lfloor 5/2 \rfloor = 2$.

1. The notation in this formula is quite cumbersome. Write out what it says for $k = 2$. For $k = 2$ it is not so bad; compute each of the symbols $\binom{k}{n}$ that appear and check that both sides of the equation agree.

2. Write out what it says for $k = 3$. In this case also, compute to verify both sides agree.

3. Write out what it says for $k = 4$.

4. Consider a class of k people. Count the number of committees of odd size as a sum of symbols $\binom{k}{n}$.

5. Consider a class of k people. Count the number of committees of even size as a sum of symbols $\binom{k}{n}$.

6. Suppose exactly one of these k people is named Sidd. Consider the following operation: given a committee, if Sidd is on it, remove him, and if Sidd is not on it, add him. Show that this transforms every even committee to an odd committee and every odd committee to an even committee.

7. Show that doing this operation twice takes you back to the original committee you started with.

8. Using the ideas above, please write a complete proof out neatly below. Have me come around and give feedback on your writeup.

2 Inclusion-Exclusion: 2 Sets

Theorem 1. *For any sets A and B,*

$$|A \cup B| = |A| + |B| - |A \cap B|.$$

We will give three proofs.

1. Imagine that the elements of $A \cup B$ are people. Each one has membership cards to the sets he/she is a member of. Any member of $A \cup B$ has one of three wallets:

   $$\{\text{A-card}\}, \quad \{\text{B-card}\}, \quad \textit{or} \quad \{\text{A-card}, \text{B-card}\},$$

   Now observe:

   (a) $|A|$ counts the people who have at least an A membership card.

   (b) $|B|$ counts the people who have at least a B membership card.

   (c) $|A \cap B|$ counts the people who have both membership cards.

   For each of these three types of people, please explain why they contribute exactly 1 to the count

   $$|A| + |B| - |A \cap B|.$$

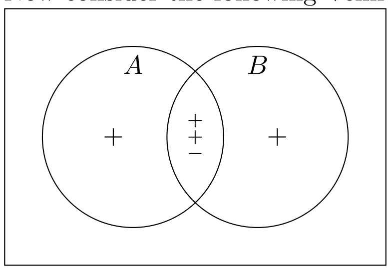
   One way to visualise or explain this is with a table. Fill out the rest of the table.

person with wallet {A-card}	$\boxed{\lvert A\rvert} + \lvert B\rvert - \lvert A \cap B\rvert$	1 + 0 - 0 = 1
person with wallet {B-card}	$\lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$	
person with wallet {A-card, B-card}	$\lvert A\rvert + \lvert B\rvert - \lvert A \cap B\rvert$	

   Now, possibly referring to the table above, write out a proof of the formula. This is a combinatorial proof: it counts the number of people in $A \cup B$ in two different ways to obtain the two sides of the equation.

2. Now we give a slightly different combinatorial proof. Imagine we have a room full of people with wallets, as before. But now we are counting the membership cards two different ways. Counting one way, we obtain $|A| + |B|$. Counting another way, we obtain $|A \cup B| + |A \cap B|$. Explain the two ways.

3. Now consider the following Venn diagram:



Explain why this picture is a proof of the statement $|A \cup B| = |A| + |B| - |A \cap B|$.

4. Which proof do you like best? What is the reason?

3 Inclusion Exclusion: 3 Sets

The goal of this section is to generalize the last theorem to three sets.

1. Determine the correct formula generalizing the last result to three sets. It should look something like

   $$|A \cup B \cup C| = |A| + ...$$

   where on the right-hand side we have just various sets and intersections of sets. Check it with me before you move on.

2. Prove it.

4 Inclusion Exclusion: n Sets

We call an intersection $A \cap B$ a *two-fold* intersection, $A \cap B \cap C$ a *three-fold* intersection, $A \cap B \cap C \cap D$ a *four-fold* intersection, and so on. We will call A by itself a *one-fold* intersection, just to match the pattern.

1. Determine the correct 'inclusion-exclusion' formula for n sets, i.e. a formula for $|A_1 \cup \cdots \cup A_n|$. If you're stuck, try $n = 4$ first. Check it with me before you move on.

2. Suppose Joe has membership cards to exactly k of the n sets. How many one-fold intersections does Joe get counted in?

3. How many two-fold intersections does Joe get counted in?

4. In general, if $1 \leq j \leq k$, how many j-fold intersections does Joe get counted in?

5. How many terms of your general inclusion-exclusion formula for $|A_1 \cup \cdots \cup A_n|$ does Joe get counted in? In other words, add up answers from the last few questions in the correct way. You don't have to evaluate or simplify the result just yet.

6. Write up a proof of your formula. May I suggest you consider the following cases:

   (a) Person with 1 card in their wallet.

   (b) Person with 2 cards in their wallet.

   (c) Person with 3 cards in their wallet.

   (d) ...

   and use previous parts of this section together with the result from Section 2.