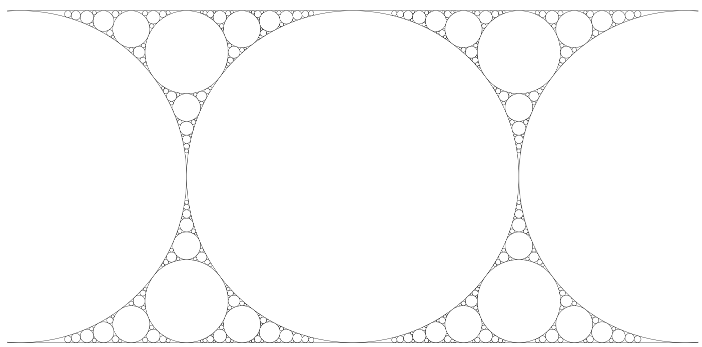
The Arithmetic of Kleinian Groups

December 1, 2015

1 Course Description

A *Kleinian group* is a discrete subgroup of $\mathrm{PSL}_2(\mathbb{C})$. Examples include

1. $\mathrm{PSL}_2(\mathbb{Z})$,
2. the Apollonian group, whose limit set is shown here:



3. the Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_K)$ where $\mathcal{O}_K$ is the ring of integers of an imaginary quadratic field K.

There are arithmetic invariants associated to a Kleinian group, specifically, the invariant trace field and invariant quaternion algebra. The purpose of the course is to the study the relationship between these arithmetic invariants and the geometry of Kleinian groups. In order to study this, we will first need to study quaternion algebras, Möbius transformations, and hyperbolic geometry.

2 Pre-requisites

For certain portions of the course, I will assume a solid working knowledge of algebraic number theory, e.g. as in Samuel's *Algebraic Theory of Numbers* and Chapter 0 of Maclachlan and Reid, including basic local theory. Some topics (in the first part of the course, including hyperbolic geometry, Möbius transformations and Kleinian groups) will be fairly independent of this prerequisite, so students lacking the prerequisite may be interested in these portions. I will also assume knowledge of other graduate pillar topics (algebra, topology, geometry, analysis) as needed.

3 Tentative Syllabus

1. Matrix groups, General and Special Linear Groups
   (a) topological groups (Beardon, §1.5)

(b) general and special linear groups

(c) metric topological structure defined by the trace (Beardon, §2.2)

(d) discrete groups (Beardon, §2.3)

(e) $SU(2, \mathbb{C})$, quaternions, conjugation isometry (Beardon, §2.4, 2.5)

2. Hyperbolic geometry in two and three dimensions (Beardon, §7)

   (a) upper half plane and upper half space models

   (b) quaternion representation

   (c) hyperbolic metric

   (d) geodesics

   (e) Lobachevski model

   (f) trigonometry

3. Complex Möbius transformations and hyperbolic isometries

   (a) Möbius transformations and basic properties, matrix representation, cross ratio

   (b) quaternions, Poincaré extension (Beardon §4.1)

   (c) elliptic, parabolic and loxodromic elements, fixed points and conjugacy classes (Beardon §4.3)

   (d) topology on Möbius group (Beardon §4.5)

   (e) reducible and elementary subgroups (Beardon §5.1)

   (f) Möbius transformations on $\mathbb{R}^n$

4. Kleinian groups

   (a) discreteness and discontinuity (Beardon §5.3)

   (b) basic properties

   (c) limit sets

   (d) cusps and stabilizers

   (e) fundamental domains and geometrical finiteness

   (f) Fuchsian groups

5. Quaternion algebras (Maclachlan and Reid, §2)

   (a) General theory, hilbert symbols and orders

   (b) norm form

   (c) orthogonal groups

   (d) clifford algebras

   (e) Splitting and ramification

(f) Over global fields

(g) Over local fields

(h) Wedderburn's Structure Theorem

(i) Skolem Noether Theorem

6. The invariant trace field and quaternion algebra of a Kleinian group (Maclachlan and Reid, §3)

7. Examples (Maclachlan and Reid, §4)

8. Applications (Maclachlan and Reid, §5)

9. Arithmetic Kleinian Groups (Machlachlan and Reid §6,7,8,11) as time permits

4 Resources

1. Maclachlan and Reid, *The arithmetic of hyperbolic* 3-*manifolds.* This is the main resource for the majority of the course. Chapter 0 will be assumed.

2. Beardon, *The geometry of discrete groups.* This is the main resource for the background in hyperbolic geometry and Kleinian groups in the first portion of the course. Chapter 1 will be assumed.

3. Samuel, *Algebraic theory of numbers.* This covers the necessary algebraic number theory background. It will be assumed.

5 Credit

Students wishing to receive credit for the course shall attend lecture regularly and also demonstrate engagement with the material. Demonstrating engagement will be in the form of one of the following:

1. Demonstrating evidence of significant homework exercises (suggested exercises will be provided), numbering at least 15. These needn't be distributed throughout the full course, thereby allowing students to concentrate on one topic (e.g. quaternion algebras, or Kleinian groups) to mastery, if that is preferred.

2. Presenting, in the latter part of the course, one of the examples or applications of Chapters 4 and 5 of Maclachlan and Reid (or similar). This will involve producing written notes to be handed out to the class.

3. Any other agreed-upon demonstration.