A TASTE OF PI
OCTOBER 16, 2010
CLOCKS, SET AND THE SECRET MATH OF SPIES

A lady of 80 named Gertie
Had a boyfriend of 60 named Bertie.
She told him emphatically
That viewed mathematically
By modulo 50, she's 30.
— John McClellan

Please work in groups of approximately 4.

1. Core Problems

For the integers modulo N, we write $\mathbb{Z}_N$. The number N is called the *modulus.* For example, $\mathbb{Z}_5$ has 5 elements, which we could write

$$\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$$

Remember that $-1 \equiv 4 \mod 5$ and $-2 \equiv 3 \mod 5$ etc.

(1) The Evil Emperor of Modulus Land (while slurping a hamster slurpee) has decided to set you a long, boring task. He says,

Complete multiplication tables for $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$, $\mathbb{Z}_5$, $\mathbb{Z}_6$, and $\mathbb{Z}_7$! Mwa-ha-ha!

He thinks it will take you at least an hour. But you are cleverer than that! Your task is to figure out how to do it in the most efficient way possible! Spread out the work among your group members. As you work, look for patterns and try to develop and share time-saving tricks for the calculations. Do you see really quick efficient ways to compute the tables? Do you see patterns in the multiplication tables that can save you time? Try to do as little calculation as possible while still getting the tables finished. (*When does your teacher ever tell you to try to do as little work as possible?!*)

(2) Make note of your time saving tricks:

Date: October 16, 2010.

$\mathbb{Z}_2$	0	1
0		
1		

$\mathbb{Z}_3$	0	1	2
0			
1			
2			

$\mathbb{Z}_4$	0	1	2	3
0				
1				
2				
3				

$\mathbb{Z}_5$	0	1	2	3	4
0					
1					
2					
3					
4					

$\mathbb{Z}_6$	0	1	2	3	4	5
0						
1						
2						
3						
4						
5						

$\mathbb{Z}_7$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							
6							

(3) Looking at your tables, can you find an element x of $\mathbb{Z}_5$ such that $2x \equiv 1 \mod 5$?

(4) Can you find an element x of $\mathbb{Z}_6$ such that $2x \equiv 1 \mod 6$?

(5) The number 1/2 isn't an integer, so we wouldn't at first think of it as part of $\mathbb{Z}_5$. But in the rational numbers, 1/2 is the number that, when multiplied with 2, gives 1. Do you think there is an element of $\mathbb{Z}_5$ which should be called '1/2'? Which one?

(6) Is there a 1/2 in $\mathbb{Z}_6$?

(7) What is $1/6 \mod 7$?

(8) What is $3/5 \mod 7$?

(9) **Conjectures.** Write down some of the patterns you notice. Can you give reasons for these patterns?

Here are some ideas:

- Try to look for patterns of the form
"When the modulus __________________________________,
then the multiplication table ______________________________".
- When does a '1/2' element exist?

(10) Graph the line $y = x+2$. In other words, put an X in every box with coordinates (x, y) that satisfies $y = x + 2$.

2					
1					
0					
4					
3					
	3	4	0	1	2

(11) Does your graph look like a line? Why or why not? How is it like the usual graph of a line? How is it different?

(12) What would the graph of $y = 2x$ look like mod 5? (Can you figure out a quick way to graph it?)

2					
1					
0					
4					
3					
	3	4	0	1	2

(13) What about $y = 2x \bmod 6$?

2						
1						
0						
5						
4						
3						
	3	4	5	0	1	2

(14) How are the lines $y = ax$ (for different a) related to the multiplication tables?

(15) How many different lines are possible modulo 5?

(16) Did you remember to count the line $x = 1$?

(17) Do you have any conjectures about lines?

(18) Next, you'll graph the relation $y^2 = x^3 + 2x + 1$. To do this, first fill in these tables:

y	y^2
0	
1	
2	
3	
4	

x	x^3	$2x$	1	$x^3 + 2x + 1$
0				
1				
2				
3				
4				

Now put the answers for the final columns (y^2 and $x^3 + 2x + 1$) next to the corresponding y and x on the axes of the graph. Finally, put an X in each box (x, y) where $y^2 = x^3 + 2x + 1$.

y^2	y						
	2						
	1						
	0						
	4						
	3						
		3	4	0	1	2	x
							$x^3 + Ax + B$

(19) Ask and answer your own questions!

2. Teasers - Pick and Choose

(*No hamsters were harmed in the writing of these puzzles.*)

(1) You are an evil hamster farmer. You have two hamster farms, and both are square plots of land with edge lengths a whole number of metres. Hamsters grow on hamster trees, and because hamsters are so cute, they use up a lot of energy to grow. Each tree takes up one square metre of space and grows one hamster per month. This month, it's your brother's birthday (he's an evil gerbil farmer), and you want to bake him hamster cupcakes. Each cupcake requires 4 hamsters. You want to have exactly three hamsters left over to make the birthday card out of. If you harvest this month's crop of hamsters, is this outcome possible?

(2) In your hamster processing facility, there's a big machine with two meshed gears that go around and around. One gear has 12 teeth and one gear has 10 teeth. The teeth are numbered, and they start with the teeth labelled 'Tooth #1' touching. How many times will the big gear turn around before this happens again?

(3) What day of the week was your birthday (your 0^{th} birthday)?

(4) At the beginning of the talk, I gave a sort of magic trick. Pick a four-digit number, with random digits. Now rearrange the digits to make another four-digit number. Subtract the smaller from the larger. Add up the digits in your answer. If the answer has more than one digit, add up the digits again. Repeat until you get one digit. Then divide by 2. The answer is 4.5. How does it work?

(5) You have a special hamster engineering facility. There are 45 genetically modified hamsters in the lab: 13 red hamsters, 15 green hamsters, and 17 blue hamsters. These hamsters have high hamster intelligence quotients (HIQs) and at night they sneak out of their cages and swap genes. Whenever two different coloured hamsters meet, they both change colour to the third colour (for example, if a green and blue hamster meet, they both turn red). Is it possible for all the hamsters in the lab to simultaneously be blue?

3. Set

You can buy a copy of SET at games stores such as Drexell Games. Here are some puzzles:

(1) If you label the colours of set with numbers, e.g. red = 0, green = 1, purple = 3, then when you have a set, what do the colours add up to? If you do the same with shadings, what do the shadings add up to? Does this give you a criteria for when three cards form a set?

(2) Is it possible to finish the game of set with exactly 3 cards left on the table?

(3) Two cards determine a SET. This is like saying two points determine a line. Do two lines determine a point? Can you find three parallel lines? What's the biggest number of parallel lines you can find?

(4) How many SETs are possible in the game?

(5) Can you make a 'magic square' of SET cards? In a magic square, each line must be a SET. Start with any three cards and try to make such a square. Can you come up with a general method?

(6) If a SET is a line in a four-dimensional space over $\mathbb{Z}_3$, then what is a plane?

(7) Ask and answer your own questions!

(8) What is the largest possible number of cards that could be showing on the table without a SET in it? This is called a 'maximal cap' and you can learn about it at http://www.warwick.ac.uk/staff/D.Maclagan/papers/set.pdf. There are lots of unanswered questions about maximal caps and related idea. (Solving those questions won't get you onto Oprah, though!)

4. Contact me!

I'll be happy to correspond with you about these problems or anything else – just email katestange@gmail.com. If you like this kind of math, I can recommend some books to check out.